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Upper and lower limits on the number of bound states in a central potential

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Abstract

In a recent paper new upper and lower limits were given, in the context of the Schrödinger or Klein–Gordon equations, for the number N_0 of S-wave bound states possessed by a monotonically nondecreasing central potential vanishing at infinity. In this paper these results are extended to the number N_ℓ of bound states for the ℓth partial wave, and results are also obtained for potentials that are not monotonic and even somewhere positive. New results are also obtained for the case treated previously, including the remarkably neat *lower* limit $N_\ell \geqslant \{\{[\sigma/(2\ell+1)+1]/2\}\}$ with $\sigma = (2/\pi)\max_{0\leqslant r<\infty}[r|V(r)|^{1/2}]$ (valid in the Schrödinger case, for a class of potentials that includes the monotonically nondecreasing ones), entailing the following *lower* limit for the total number N of bound states possessed by a monotonically nondecreasing central potential vanishing at infinity: $N \geqslant \{\{(\sigma+1)/2\}\}\{\{(\sigma+3)/2\}\}/2$ (here the double braces denote the integer part).

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1. Introduction and main results

In a previous paper [4] new upper and lower limits were provided for the number N_0 of S-wave bound states possessed, in the framework of the Schrödinger or Klein–Gordon equations, by a central potential $V(r)$ vanishing at infinity and having the property to yield a nowhere repulsive force, so that, for all (non-negative) values of the radius r,

$$V'(r) \geqslant 0 \tag{1}$$

hence

$$-V(r) = |V(r)|. \tag{2}$$

In (1), and always below, the appended primes signify differentiation with respect to the radius r. The main purpose of this paper is to extend the results of [4] to higher partial waves, namely to provide *new* upper and lower limits for the number N_ℓ of ℓ-wave bound states possessed by a central potential $V(r)$. Hereafter, ℓ is the angular momentum quantum number (a non-negative integer). For simplicity we restrict attention here to the Schrödinger case, since the extension of the results to the Klein–Gordon case is essentially trivial, see [4]. As in [4] we assume the potential to be finite for $0 < r < \infty$, to vanish at infinity faster than the inverse square of r,

$$\lim_{r\to\infty}[r^{2+\varepsilon}V(r)] = 0 \tag{3}$$

and, unless otherwise specified, not to diverge at the origin faster than the inverse square of r,

$$\lim_{r\to 0}[r^{2-\varepsilon}V(r)] = 0. \tag{4}$$

Here ε denotes some positive quantity, $\varepsilon > 0$. Moreover, in the following the 'monotonicity' property (1) is generally replaced by the less stringent condition

$$-\frac{V'(r)}{V(r)} + \frac{4\ell}{r} \geqslant 0 \tag{5a}$$

or equivalently

$$[V(r)r^{-4\ell}]' \geqslant 0 \tag{5b}$$

which is automatically satisfied by monotonic potentials vanishing at infinity, see (1) and (2); and we also obtain results for potentials that do not necessarily satisfy for all values of r the 'monotonicity' condition (5) and possibly not even the 'negativity' property (2). In any case the properties of the potential required for the validity of the various results reported below will be specified in each case.

In the process of deriving the results presented below we also uncovered some *new* neat limits (such as those reported in the abstract) which are as well applicable in the S-wave case and are different from those given in [4]. These results therefore extend those presented in [4].

From the upper and lower limits for N_ℓ one can obtain the upper and lower limits for the total number,

$$N = \sum_{\ell=0}^{L}(2\ell+1)N_\ell \tag{6}$$

of bound states possessed by the potential $V(r)$; the upper limit, L, of the sum on the right-hand side of formula (6) is the largest value of ℓ for which the potential $V(r)$ possesses bound states. It is well known that conditions (3) and (4) are sufficient to guarantee that both L and N are finite. *New* upper and lower limits on the values of the maximal angular momentum quantum number L for which bound states do exist are also shown below, as well as *new* upper and lower limits on the total number of bound states N. Note that we are assuming, see (6), working in the (ordinary) three-dimensional world, with spherically symmetrical potentials.

As in [4], we begin below with a terse review of *known* results, and we then show our *new* upper and lower limits and briefly outline their main features. A more detailed discussion of the properties of these *new* limits, including tests for various potentials of their cogency (compared with that of previously known limits), is then presented in section 2. The proofs of our results are given in section 3, and some final remarks in section 4.

1.1. Units and preliminaries

We use the standard quantum mechanical units such that $2m = \hbar = 1$, where m is the mass of the particle bound by the central potential $V(r)$. This entails that the potential $V(r)$ has the dimension of an inverse square length, hence the following two quantities are dimensionless:

$$S = \frac{2}{\pi}\int_0^\infty \mathrm{d}r[-V^{(-)}(r)]^{1/2} \tag{7}$$

$$\sigma = \frac{2}{\pi}\max_{0\leqslant r<\infty}\{r[-V^{(-)}(r)]^{1/2}\}. \tag{8}$$

Hereafter, $V^{(-)}(r)$ denotes the potential that is obtained from $V(r)$ by setting its positive part to zero,

$$V^{(-)}(r) = V(r)\theta[-V(r)]. \tag{9}$$

Hereafter, $\theta(x)$ is the standard step function, $\theta(x) = 1$ if $x \geqslant 0$, $\theta(x) = 0$ if $x < 0$.

These quantities, S and σ, play an important role in the following. The motivation for inserting the $(2/\pi)$ prefactor in these definitions is to make neater some of the formulae given below.

As for the ℓ-wave radial Schrödinger equation, in these standard units it reads

$$u_\ell''(\kappa;r) = \left[\kappa^2 + V(r) + \frac{\ell(\ell+1)}{r^2}\right]u_\ell(\kappa;r). \tag{10}$$

The eigenvalue problem based on this ordinary differential equation (ODE) characterizes the (moduli of the) ℓ-wave bound-state energies, $\kappa^2 = \kappa_{\ell,n}^2$, via the requirement that the corresponding eigenfunctions, $u_\ell(\kappa_{\ell,n};r)$ vanish at the origin,

$$u_\ell(\kappa_{\ell,n};0) = 0 \tag{11}$$

and be normalizable, hence vanish at infinity,

$$\lim_{r\to\infty}[u_\ell(\kappa_{\ell,n};r)] = 0. \tag{12}$$

It is well known that conditions (3) and (4) on the potential $V(r)$ are sufficient to guarantee that the (singular) Sturm–Liouville problem characterized by the ODE (10) with the boundary conditions (11) and (12) has a finite (possibly vanishing) number of discrete eigenvalues $\kappa_{\ell,n}^2$. To count them one notes that for sufficiently large (for definiteness, positive) values of κ the solution $u_\ell(\kappa;r)$ of the radial Schrödinger equation (10) with the boundary condition (11) (which characterizes the solution uniquely up to a multiplicative constant) has no zeros in the interval $0 < r < \infty$ and diverges as $r \to \infty$ (proportionally to $\exp(\kappa r)$), because for sufficiently large values of κ the quantity in the square brackets on the right-hand side of the radial Schrödinger equation (10) is positive for all values of r, hence the solution $u_\ell(\kappa;r)$ of the second-order ODE (10) is everywhere convex. Let us then imagine decreasing gradually the value of the positive constant κ so that the quantity in the square brackets on the right-hand side of (10) becomes negative in some region(s) (for this to happen the potential $V(r)$ must itself be negative in some region(s), this being a necessary condition for the existence of bound states), entailing that the solution $u_\ell(\kappa;r)$ becomes concave in that region(s). For some value, say $\kappa = \kappa_{\ell,1}$, the solution $u_\ell(\kappa_{\ell,1};r)$ may then have a zero at $r = \infty$, namely vanish as $r \to \infty$ (proportionally to $\exp(-\kappa_{\ell,1}r)$), thereby satisfying the boundary condition at infinity (12) hence qualifying as a bound-state wavefunction and thereby entailing that $\kappa_{\ell,1}^2$ is the (modulus of the) binding energy of the first (the most bound) ℓ-wave state associated with the potential $V(r)$. If one decreases κ below $\kappa_{\ell,1}$, the zero will then enter (from the right) the interval $0 < r < \infty$, occurring, say, at $r = r_{\ell,1}(\kappa)$ (namely $u_\ell[\kappa; r_{\ell,1}(\kappa)] = 0$ with $0 < r_{\ell,1}(\kappa) < \infty$), since the effect of decreasing κ, by decreasing the value of the quantity in the square brackets on the right-hand side of the radial Schrödinger equation (10), is to make the solution $u_\ell(\kappa; r)$ more concave, hence to move its zeros to smaller values of r (towards the left on the positive real line $0 < r < \infty$). Continuing the process of decreasing κ, for $\kappa = \kappa_{\ell,2}$ a second zero of the solution $u_\ell(\kappa; r)$ may appear at $r = \infty$, entailing that this solution, $u_\ell(\kappa_{\ell,2}; r)$, again satisfies the boundary condition at infinity (12), hence qualifies as a bound-state wavefunction, implying that $\kappa_{\ell,2}^2$ is the (modulus of the) binding energy of the next most bound ℓ-wave state associated with the potential $V(r)$. The process can then be continued, yielding a sequence of decreasing (in modulus) binding energies $\kappa_{\ell,n}^2$ with $n = 1, 2, \ldots, N_\ell$. Correspondingly, the solution $u_\ell(\kappa; r)$ of the radial Schrödinger equation (10) characterized by the boundary condition (11) shall have, for $\kappa_{\ell,n-1} > \kappa > \kappa_{\ell,n}$, $n - 1$ zeros in the interval $0 < r < \infty$. The process of decreasing the parameter κ we just described ends when this parameter reaches the value $\kappa = 0$, and it clearly entails that the number of zeros $r_{\ell,n}(0)$ of the zero-energy solution $u_\ell(0; r)$ of the radial Schrödinger equation (10) characterized by the boundary condition (11) coincides with the number N_ℓ of bound states possessed by the potential $V(r)$ (namely, $u_\ell[0; r_{\ell,n}(0)] = 0$ with $0 < r_{\ell,1} < r_{\ell,2} < \cdots < r_{\ell,N_\ell-1} < r_{\ell,N_\ell} < \infty$).

Hence in the following, as indeed in [4], in order to obtain upper and lower limits on the number N_ℓ of ℓ-wave bound states we focus on obtaining upper and lower limits on the number N_ℓ of zeros of the zero-energy solution $u_\ell(0; r)$ of the radial Schrödinger equation (10) characterized by the boundary condition (11) (for notational simplicity, these zeros will be hereafter denoted as z_n, and the zero-energy solution of the radial Schrödinger equation (10) characterized by the boundary condition (11) as $u(r)$, namely $u(r) \equiv u_\ell(0; r)$ with $u(z_n) = 0, 0 < z_1 < \cdots < z_{N_\ell} < \infty$, see section 3). Let us moreover emphasize that here, and throughout this paper, we ignore the marginal possibility that the potential $V(r)$ under consideration possesses a 'zero-energy' bound state, namely that $u(r)$ vanishes as $r \to \infty$ (for $\ell > 0$), or tends to a constant value in the S-wave case; namely, we assume $z_{N_\ell} < \infty$, because taking this possibility into account would force us to go several times into cumbersome details, the effort to do so being clearly out of proportion to the additional clarification gained.

In the following subsections we briefly review the *known* expressions, in terms of a given central potential $V(r)$, of upper and lower limits on the number N_ℓ of ℓ-wave bound states, and also on the maximum value, L, of the angular momentum quantum number for which bound states do exist, as well as on the total number N of bound states, see (6); and we also present our *new* upper and lower limits on these quantities.

Before listing these upper and lower limits let us note that an immediate hunch on the accuracy of these limits for strong potentials may be obtained via the introduction of a (dimensionless, positive) 'coupling constant' g by setting

$$V(r) = g^2 v(r) \tag{13}$$

where $v(r)$ is assumed to be independent of g, and by recalling that, at large g, N_ℓ increases proportionally to g [7],

$$N_\ell \sim g \qquad \text{as} \quad g \to \infty \tag{14a}$$

indeed [9]

$$N_\ell \approx \frac{1}{\pi} \int_0^\infty \mathrm{d}r[-V^{(-)}(r)]^{1/2} \qquad \text{as} \quad g \to \infty. \tag{14b}$$

Hereafter, we denote asymptotic equality, respectively proportionality, with the symbols $\approx$, respectively $\sim$.

The analogous asymptotic behaviour of L and N reads

$$L \sim g \qquad \text{as} \quad g \to \infty \tag{15a}$$

indeed [23]

$$L \approx \max_{0\leqslant r<\infty} \{r[-V^{(-)}(r)]^{1/2}\} \qquad \text{as} \quad g \to \infty \tag{15b}$$

and

$$N \sim g^3 \qquad \text{as} \quad g \to \infty \tag{16a}$$

indeed [19]

$$N \approx \frac{2}{3\pi} \int_0^\infty \mathrm{d}r\, r^2[-V^{(-)}(r)]^{3/2} \qquad \text{as} \quad g \to \infty. \tag{16b}$$

1.2. Limits defined in terms of global properties of the potential (i.e., involving integrals over the potential)

In this subsection we only consider results which can be formulated in terms of integrals over the potential $V(r)$, possibly raised to a power, see below. Firstly, we tersely review *known* upper and lower limits on the number N_ℓ of bound states, as well as *known* upper and lower limits on the maximum value L for which bound states do exist, and then provide *new* upper and lower limits for both N_ℓ and L.

The earliest upper limit of this kind on the number N_ℓ of ℓ-wave bound states is due to Bargmann [1] (and then also discussed by Schwinger [22]), and we hereafter refer to it as the BSℓ upper limit:

$$\text{BS}\ell\text{:} \quad N_\ell < \frac{1}{2\ell+1} \int_0^\infty \mathrm{d}r\, r[-V^{(-)}(r)]. \tag{17}$$

Remark. In writing this upper limit we have used the strict inequality sign; *we will follow this rule in all the analogous formulae we write hereafter*. Let us repeat that in this manner we systematically ignore the possibility that a potential possesses exactly the number of bound states given by the (upper or lower) limit expression being displayed (which in such a case would have to yield an integer value), since this would correspond to the occurrence of a 'zero-energy bound state' (in the S-wave case) or a 'zero-energy resonance' (in the higher-wave case)—a marginal possibility which we believe can be ignored without significant loss of generality.

Since the right-hand side of inequality (17) increases proportionally to g^2 (see (13)) rather than g (see (14)) as g diverges, for strong potentials possessing many bound states the upper limit (17) is generally very far from the exact value. This clearly implies the following upper limit on L:

$$\text{BSL:} \quad L < L_{\text{BSL}}^{(+)} = -\frac{1}{2} + \frac{1}{2} \int_0^\infty \mathrm{d}r\, r[-V^{(-)}(r)]. \tag{18}$$

The right-hand side of this inequality also increases proportionally to g^2 (see (13)) rather than to g (see (15)) as g diverges, hence for strong potentials possessing many bound states the upper limit (18) is also generally very far from the exact value. The limit BSℓ is however the best possible, namely there is a potential $V(r)$,

$$V(r) = gR^{-1} \sum_{n=1}^{N_\ell} \alpha_n\, \delta(r - \beta_n R) \tag{19}$$

(with an appropriate assignment of the $2N_\ell$ dimensionless constants α_n and β_n, depending on g, ℓ and N_ℓ) that possesses N_ℓ ℓ-wave bound states and for which the right-hand side of (17) takes a value arbitrarily close to N_ℓ.

The next upper limit we report is due to Chadan *et al* [11], and we denote it as CMS. It holds only for potentials that satisfy the monotonicity condition (1), and reads (see (7))

$$\text{CMS}: \quad N_\ell < S + 1 - \sqrt{1 + \left(\frac{2}{\pi}\right)^2 \ell(\ell+1)}. \tag{20}$$

A less stringent but neater version [11] of this upper limit, which we denote as CMSn, reads

$$\text{CMSn}: \quad N_\ell < S + 1 - \frac{2\ell+1}{\pi}. \tag{21}$$

Clearly inequality (21) entails the following neat upper limit on L, which we denote as CMSL:

$$\text{CMSL}: \quad L < L^{(+)}_{\text{CMSL}} = \frac{\pi}{2} S - \frac{1}{2}. \tag{22}$$

A more stringent but less neat upper limit on L, which we do not write, can be obtained from the CMS upper limit (20).

The next upper limit we report is immediately implied by a result due to Martin [20], and we denote it as Mℓ. It reads

$$\text{M}\,\ell: \quad N_\ell < \left[\int_0^\infty \mathrm{d}r\, r^2 V^{(-)}_{\ell,\text{eff}}(r) \int_0^\infty \mathrm{d}r\, V^{(-)}_{\ell,\text{eff}}(r)\right]^{1/4} \tag{23}$$

with $V^{(-)}_{\ell,\text{eff}}(r)$ being the negative part of the 'effective ℓ-wave potential' (see (10)):

$$V_{\ell,\text{eff}}(r) = V(r) + \frac{\ell(\ell+1)}{r^2}. \tag{24}$$

Finally, the last two upper limits of this type we report are due to Glaser *et al* [13], and again to Chadan *et al* [12], and we denote them as GGMT and CMS2. The first of these upper limits reads

$$\text{GGMT}: \quad N_\ell < (2\ell+1)^{1-2p} C_p \int_0^\infty \frac{\mathrm{d}r}{r} [-r^2 V^{(-)}(r)]^p \tag{25a}$$

with

$$C_p = \frac{(p-1)^{p-1}\Gamma(2p)}{p^p \Gamma^2(p)} \tag{25b}$$

and the restriction $p \geqslant 1$. The upper limit GGMT is however always characterized by an unsatisfactory dependence on g as $g \to \infty$ (see (13)): the right-hand side of (25*a*) is proportional to g^{2p} with $p \geqslant 1$ rather than to g (see (13) and (14)), hence it always yields a result far from the exact value for strong potentials possessing many bound states.

The second of these upper limits reads

$$\text{CMS2}: \quad N_\ell < (2\ell+1)^{1-2p} \tilde{C}_p \int_0^\infty \frac{\mathrm{d}r}{r} |r^2 V(r)|^p \tag{26a}$$

with

$$\tilde{C}_p = p(1-p)^{p-1} \tag{26b}$$

with the restriction $1/2 \leqslant p < 1$, and it is valid provided the potential is nowhere positive, see (2), and moreover satisfies for all values of r, $0 \leqslant r < \infty$, the relation

$$\{r^{1-2p}[-V(r)]^{1-p}\}' \leqslant 0. \tag{27}$$

Note that the right-hand side of (26*a*) features the correct power growth proportional to g, see (13) and (14), only if $p = 1/2$, in which case condition (27) is equivalent to condition (1) but the upper limit CMS2 then reads (see (25) and (7))

$$N_\ell < 2^{-3/2}\pi S \tag{28}$$

hence it is analogous, but less stringent (since $2^{-3/2}\pi \cong 1.11 > 1$), than the limit CC,

$$\text{CC:}\quad N_\ell < S \tag{29}$$

(see equation (1.4) of [4]; this limit, valid for S-waves, is *a fortiori* valid for all partial waves—albeit clearly not very good for large ℓ, and moreover off by a factor of 2 in the asymptotic limit of strong potentials, see (13) and (14*b*)—indeed the main motivation for, and achievement of, the research reported in [4] was just to provide upper and lower limits to N_0 that do not have this last defect).

Let us now turn to *known* lower limits, always restricting our consideration here to results which can be formulated in terms of integrals over the potential $V(r)$, possibly raised to a power, see below.

Only one result of this kind seems to be previously known [6, 7], and we will denote it as Cℓ. It states (without requiring any additional conditions on the potential other than (3) and (4)—and even these conditions are sufficient but not necessary) that

$$\text{C}\ell\text{:}\quad N_\ell > -\frac{1}{2} + \frac{1}{\pi}\max_{0\leqslant a<\infty}\left\{\int_0^\infty \mathrm{d}r\,\min\left[a^{-1}\left(\frac{r}{a}\right)^{2\ell}, -aV(r)\left(\frac{r}{a}\right)^{-2\ell}\right]\right\}. \tag{30}$$

In this formula, and hereafter, the notation min$[x, y]$ signifies x if $x \leqslant y$, y if $y \leqslant x$. Let us now assume that the equation

$$a^{-1}\left(\frac{r}{a}\right)^{2\ell} = -aV(r)\left(\frac{r}{a}\right)^{-2\ell} \tag{31}$$

admits one and only one solution, say $r = R(a)$ (and note that validity for all values of r of the 'monotonicity condition' (5) is sufficient to guarantee that this is indeed the case), so that the lower limit (30) can be rewritten as follows:

$$N_\ell > -\frac{1}{2} + \frac{1}{\pi}\max_{0\leqslant a<\infty}\left\{\int_0^{R(a)} \mathrm{d}r\,a^{-1}\left(\frac{r}{a}\right)^{2\ell} - \int_{R(a)}^\infty \mathrm{d}r\,aV(r)\left(\frac{r}{a}\right)^{-2\ell}\right\} \tag{32}$$

where $r = R(a)$ is the solution of (31). It is then easy, using (31), to calculate the maximum of the right-hand side of this inequality (note that the first integral inside the braces on the right-hand side of the above inequality, (32), is elementary) and to obtain thereby the following lower limit, that we denote here as Cℓn:

$$\text{C}\ell\text{n:}\quad N_\ell > -\frac{1}{2} + \frac{2}{\pi}\frac{\rho|V(\rho)|^{1/2}}{2\ell+1} \tag{33a}$$

with the radius ρ defined as the solution of the following equation:

$$\rho V(\rho) = (2\ell+1)\int_\rho^\infty \mathrm{d}r\left(\frac{\rho}{r}\right)^{2\ell} V(r). \tag{33b}$$

The lower limit Cℓn presents the correct dependence on g, see (13) and (14), since clearly ρ does not depend on g. This limit is the best possible, and the potential that saturates it has the form [6, 7]

$$V(r) = -g^2R^{-2}\left(\frac{r}{R}\right)^{4\ell} \qquad \text{for}\quad 0 \leqslant r < \alpha R \tag{34a}$$

$$V(r) = 0 \qquad \text{for}\quad r \geqslant \alpha R \tag{34b}$$

with R being an arbitrary (*positive*) radius and α a dimensionless constant given by

$$\alpha = \left[\frac{\pi(2\ell+1)(N_\ell+\delta)}{g}\right]^{1/(2\ell+1)} \qquad 0 \leqslant \delta < 1/2. \tag{35}$$

Let us now present our *new* upper and lower limits on the number of bound states possessed by the central potential $V(r)$. All these results are proved in section 3.

We begin with a *new* upper limit on the number N_0 of S-wave bound states, possessed by a central potential $V(r)$ that features the following properties: it has two zeros, $V(r_\pm) = 0$ (with $r_- < r_+$), is positive for r smaller than r_-, negative for r in the interval from r_- to r_+, and again positive for r larger than r_+,

$$V(r) > 0 \qquad \text{for} \quad 0 \leqslant r < r_- \tag{36a}$$

$$V(r) < 0 \qquad \text{for} \quad r_- < r < r_+ \tag{36b}$$

$$V(r) > 0 \qquad \text{for} \quad r_+ < r < \infty. \tag{36c}$$

Note that we do not exclude the possibility that the potential diverges (but then to *positive* infinity) at the origin; so, for the validity of the result we now report, condition (4) need not hold, and indeed even condition (3) can be forsaken, provided the potential does vanish at infinity, $V(\infty) = 0$. Indeed one option we shall exploit below is to replace the potential $V(r)$ with $V_{\ell,\mathrm{eff}}(r)$, see (24), and to thereby include in the present framework the treatment of the ℓ-wave case. On the other hand the assumption that the potential has only two zeros and no more is made here for simplicity; the extension to potentials having more than two zeros is straightforward, but the corresponding results lack the neatness that justifies their explicit presentation here (we trust any potential user of our results who needs to apply them to the more general case of a potential with more than two zeros will be able to obtain easily the relevant formulae by extending the treatment of section 3). Let us also note that the following results remain valid (but may become trivial) if $r_- = 0$ or $r_+ = \infty$.

We denote as NUL1 ('new upper limit no 1') the following result:

$$\text{NUL1}: \quad N_0 < 1 + \frac{2}{\pi}\left\{(r_+ - r_-)\int_0^\infty \mathrm{d}r[-V^{(-)}(r)]\right\}^{1/2}. \tag{37}$$

It can actually be shown (see subsection 3.7) that the upper limit NUL1 is generally less cogent than the upper limit NUL2, see (44), but it has the advantage over NUL2 of being simpler, and for this reason it is nevertheless worthwhile to report it separately here.

Let us now report a *new* lower limit on the number N_0 of S-wave bound states that holds for potentials that satisfy the same conditions (36), and which we denote as NLL1 ('new lower limit no 1'). It is actually a variation of the lower limit Cℓ, see (30), and reads

$$\text{NLL1}: \quad N_0 > -1 + \frac{1}{\pi}\max_{0\leqslant a<\infty}\left\{\int_0^\infty \mathrm{d}r\,\min(a^{-1}, a[-V^{(-)}(r)])\right\}. \tag{38}$$

A less cogent but perhaps simpler version of this lower bound reads

$$\text{NLL1n}: \quad N_0 > -1 + \frac{1}{\pi}\left\{\int_0^\infty \mathrm{d}r[-V^{(-)}(r)]\right\}\{\max[-V^{(-)}(r)]\}^{-1/2} \tag{39}$$

(it is clearly obtained from NLL1 by setting $a = \{\max[-V^{(-)}(r)]\}^{-1/2}$).

These *new* upper and lower limits become relevant to the number N_ℓ of ℓ-wave bound states possessed by the central potential $V(r)$ via the replacement in the above inequalities, (37) and (39), of $V(r)$ with $V_{\ell,\mathrm{eff}}(r)$, see (24). Note that, for a large class of central potentials

$V(r)$ satisfying conditions (3) and (4), the effective ℓ-wave potential $V_{\ell,\text{eff}}(r)$, especially for $\ell > 0$, is indeed likely to satisfy the conditions (see (36))

$$V_{\ell,\text{eff}}(r) = V(r) + \frac{\ell(\ell+1)}{r^2} > 0 \qquad \text{for} \quad 0 \leqslant r < r_-^{(\ell)} \tag{40a}$$

$$V_{\ell,\text{eff}}(r) = V(r) + \frac{\ell(\ell+1)}{r^2} < 0 \qquad \text{for} \quad r_-^{(\ell)} < r < r_+^{(\ell)} \tag{40b}$$

$$V_{\ell,\text{eff}}(r) = V(r) + \frac{\ell(\ell+1)}{r^2} > 0 \qquad \text{for} \quad r_+^{(\ell)} < r < \infty \tag{40c}$$

required for the validity of the upper and lower limits NUL1 and NLL1, see (37) and (38). We denote the *new* upper and lower limits obtained in this manner as NUL1ℓ and NLL1nℓ:

$$\text{NUL1}\ell: \quad N_\ell < 1 + \frac{2}{\pi}\left\{\left(r_+^{(\ell)} - r_-^{(\ell)}\right)\left[-\ell(\ell+1)\left(\frac{1}{r_-^{(\ell)}} - \frac{1}{r_+^{(\ell)}}\right) + \int_{r_-^{(\ell)}}^{r_+^{(\ell)}} \mathrm{d}r |V(r)|\right]\right\}^{1/2} \tag{41}$$

$$\text{NLL1n}\ell: \quad N_\ell > -1 + \frac{1}{\pi}\left[-\ell(\ell+1)\left(\frac{1}{r_-^{(\ell)}} - \frac{1}{r_+^{(\ell)}}\right) + \int_{r_-^{(\ell)}}^{r_+^{(\ell)}} \mathrm{d}r |V(r)|\right]$$
$$\times \left[\max_{r_-^{(\ell)}<r<r_+^{(\ell)}} \left|V(r) + \frac{\ell(\ell+1)}{r^2}\right|\right]^{-1/2}. \tag{42}$$

Next, we report *new* upper and lower limits on the number N_0 of S-wave bound states somewhat analogous to those given in [4], but applicable to nonmonotonic potentials. As above, we restrict for simplicity our consideration to potentials that satisfy conditions (36). We do moreover, again for simplicity, require the potential $V(r)$ to possess only one minimum, at $r = r_{\min}$:

$$V(r) > 0 \qquad \text{for} \quad 0 \leqslant r < r_- \tag{43a}$$

$$V(r) < 0,\ V'(r) \leqslant 0 \qquad \text{for} \quad r_- < r \leqslant r_{\min} \tag{43b}$$

$$V(r) < 0,\ V'(r) \geqslant 0 \qquad \text{for} \quad r_{\min} \leqslant r < r_+ \tag{43c}$$

$$V(r) > 0 \qquad \text{for} \quad r_+ < r < \infty. \tag{43d}$$

We denote these *new* upper, respectively lower, limits on the number N_0 of S-wave bound states as NUL2, respectively NLL2:

$$\text{NUL2}: \quad N_0 < 1 + \frac{S}{2} + \frac{1}{2\pi}\log\left[\frac{-V^{(-)}(r_{\min})}{M}\right] \tag{44}$$

$$\text{NLL2}: \quad N_0 > -\frac{3}{2} + \frac{S}{2} - \frac{1}{2\pi}\log\left[\frac{-V^{(-)}(r_{\min})}{M}\right] \tag{45}$$

where S is defined by (7) and

$$M = \min[-V^{(-)}(p), -V^{(-)}(q)] \tag{46}$$

with the two radii p and q defined as the solutions of the following equations:

$$\int_0^p \mathrm{d}r[-V^{(-)}(r)]^{1/2} = \frac{\pi}{2} \tag{47}$$

$$\int_q^{\infty} \mathrm{d}r[-V^{(-)}(r)]^{1/2} = \frac{\pi}{2} \tag{48}$$

and with the additional condition (which might rule out the applicability of these limits to potentials possessing very few bound states, but which is certainly satisfied by potentials that are sufficiently strong to possess several bound states)

$$p \leqslant r_{\min} \leqslant q. \tag{49}$$

As already mentioned above and explained in section 3.7, the upper limit NUL2, see (44), is generally more cogent than the upper limit NUL1, see (37), but it requires the additional computation of the two radii p and q.

Again, as above, *new* limits (hereafter denoted NUL2ℓ, respectively NLL2ℓ) on the number N_ℓ of ℓ-wave bound states possessed by the central potential $V(r)$ are entailed by these results via the replacement of $V(r)$ with $V_{\ell,\mathrm{eff}}(r)$, see (24), so that the relevant formulae read as follows:

$$V_{\ell,\mathrm{eff}}(r) > 0 \qquad \text{for} \quad 0 \leqslant r < r_-^{(\ell)} \tag{50a}$$

$$V_{\ell,\mathrm{eff}}(r) < 0,\ V'_{\ell,\mathrm{eff}}(r) \leqslant 0 \qquad \text{for} \quad r_-^{(\ell)} < r \leqslant r_{\min}^{(\ell)} \tag{50b}$$

$$V_{\ell,\mathrm{eff}}(r) < 0,\ V'_{\ell,\mathrm{eff}}(r) \geqslant 0 \qquad \text{for} \quad r_{\min}^{(\ell)} \leqslant r < r_+^{(\ell)} \tag{50c}$$

$$V_{\ell,\mathrm{eff}}(r) > 0 \qquad \text{for} \quad r_+^{(\ell)} < r < \infty \tag{50d}$$

$$\text{NUL2}\,\ell\!: \quad N_\ell < 1 + \frac{1}{\pi}\int_{r_-^{(\ell)}}^{r_+^{(\ell)}} \mathrm{d}r\big[-V_{\ell,\mathrm{eff}}^{(-)}(r)\big]^{1/2} + \frac{1}{2\pi}\log\left\{\frac{\big[-V_{\ell,\mathrm{eff}}^{(-)}\big(r_{\min}^{(\ell)}\big)\big]}{M}\right\} \tag{51}$$

$$\text{NLL2}\,\ell\!: \quad N_\ell > -\frac{3}{2} + \frac{1}{\pi}\int_{r_-^{(\ell)}}^{r_+^{(\ell)}} \mathrm{d}r\big[-V_{\ell,\mathrm{eff}}^{(-)}(r)\big]^{1/2} - \frac{1}{2\pi}\log\left\{\frac{\big[-V_{\ell,\mathrm{eff}}^{(-)}\big(r_{\min}^{(\ell)}\big)\big]}{M}\right\} \tag{52}$$

$$\int_{r_-^{(\ell)}}^{p^{(\ell)}} \mathrm{d}r\big[-V_{\ell,\mathrm{eff}}^{(-)}(r)\big]^{1/2} = \frac{\pi}{2} \tag{53}$$

$$\int_{q^{(\ell)}}^{r_+^{(\ell)}} \mathrm{d}r\big[-V_{\ell,\mathrm{eff}}^{(-)}(r)\big]^{1/2} = \frac{\pi}{2} \tag{54}$$

$$p^{(\ell)} \leqslant r_{\min}^{(\ell)} \leqslant q^{(\ell)} \tag{55}$$

$$M = \min\big[-V_{\ell,\mathrm{eff}}^{(-)}(p^{(\ell)}), -V_{\ell,\mathrm{eff}}^{(-)}(q^{(\ell)})\big]. \tag{56}$$

Let us now report another *new* lower limit on the number N_ℓ of ℓ-wave bound states applicable to nonmonotonic potentials. As above, we restrict for simplicity our consideration to potentials that satisfy conditions (36). We do moreover, again for simplicity, require the potential $V(r)$ to possess only one minimum, at $r = r_{\min}$, see (43). We denote it by NLL3s:

$$\text{NLL3s}\!: \quad N_\ell > -1 + \frac{1}{\pi}\int_0^s \mathrm{d}r[-V^{(-)}(r)]^{1/2} - \frac{1}{4\pi}\log\left\{\frac{[V^{(-)}(r_{\min})]^2}{V^{(-)}(p)V^{(-)}(s)}\right\} - \frac{\ell}{\pi}\log\left(\frac{s}{p}\right) \tag{57}$$

where the radius p is defined by (47) and s is an arbitrary radius (larger than p, $s > p$). The value of s that yields the most stringent limit is a (or the) solution of the equation

$$sV'(s) = 4s|V(s)|^{3/2} + 4\ell V(s) \tag{58}$$

(since clearly for this value of s the potential $V(r)$ is negative, $V(s) = -|V(s)|$, in this formula $V(s)$ could be replaced by $V^{(-)}(s)$ with the condition $r_- < s < r_+$, see (36)).

A neater, if marginally less stringent, version of the lower limit NLL3s, which we denote as NLL3, reads as follows:

$$\text{NLL3}: \quad N_\ell > \nu - \frac{\ell}{\pi} \log\left(\frac{q}{p}\right) \tag{59a}$$

where

$$\nu = -\frac{3}{2} + \frac{1}{2}S - \frac{1}{4\pi} \log\left\{\frac{[V^{(-)}(r_{\min})]^2}{V^{(-)}(p)V^{(-)}(q)}\right\}. \tag{59b}$$

Here p is defined as above, see (47), while q is defined by formula (48), and S is defined by (7). Here and below, see (59) and (60), as well as in all subsequent formulae involving both p and q, see (47) and (48), we always assume validity of the inequality $q \geqslant p$, as is indeed generally the case for any potential possessing enough bound states. (These results are also valid if the potential has only one zero or no zero at all, and even if the derivative of the potential never vanishes; in the latter case $r_{\min}$ in (57) and (59b) must be replaced by p.)

Clearly the lower limit NLL3, see (59), implies the following *new* lower limit $L^{(-)}_{\text{NLL3L}}$ on the largest value L of the angular momentum quantum number ℓ for which the potential $V(r)$ possesses bound states (entailing that for $\ell \leqslant L^{(-)}_{\text{NLL3L}}$ the potential $V(r)$ does certainly possess at least one ℓ-wave bound state):

$$\text{NLL3L}: \quad L \geqslant L^{(-)}_{\text{NLL3L}} = \left\{\left\{\pi\left[\log\left(\frac{q}{p}\right)\right]^{-1} \nu\right\}\right\} \tag{60}$$

with p, q and ν being defined by (47), (48) and (59b). Here the double braces denote the integer part.

1.3. Limits defined in terms of local properties of the potential (not involving integrals over the potential)

In this subsection we report a *new* lower limit on the number of ℓ-wave bound states N_ℓ, which depends on the potential only via the quantity σ, see (8). Note that, perhaps with a slight abuse of language, we consider (see the title of this section) the quantity σ to depend only on the local properties of the potential, since to calculate it only the value(s) of r at which the function $2V(r) + rV'(r)$ vanishes must be identified. We also provide, in terms of the quantity σ, *new* upper and lower limits on the largest value L of ℓ for which the potential $V(r)$ possesses bound states.

The lower limit on N_ℓ, which we denote NLL4, holds provided the potential $V(r)$ satisfies, for all values of r, $0 \leqslant r < \infty$, the inequality (5), which as we already noted above, is automatically satisfied by monotonically nondecreasing potentials, see (1). It takes the neat form

$$\text{NLL4}: \quad N_\ell > -\frac{1}{2} + \frac{\sigma}{2(2\ell+1)}. \tag{61}$$

This lower limit features the correct power growth, see (14a), as g (see (13)) diverges, and it is the best possible, being saturated by the potential (34) with (35). The analogy of the lower limit NLL4, see (61), with the lower limit Cℓn, see (33), is remarkable; note that, since obviously $\sigma \geqslant (2/\pi)\rho|V(\rho)|^{1/2}$ (see (8)), the *new* limit NLL4, would always be more stringent than Cℓn, were it not for the additional factor $1/2$ multiplying σ on the right-hand side of the inequality (61) (in comparison with (33*a*)).

The result (61) clearly entails the following *new* lower limit $L^{(-)}_{\mathrm{NLL4L}}$ on the largest value L of ℓ for which the potential $V(r)$ possesses bound states (entailing that for $\ell \leqslant L^{(-)}_{\mathrm{NLL4L}}$ the potential $V(r)$ does certainly possess at least one ℓ-wave bound state):

$$\text{NLL4L}: \quad L \geqslant L^{(-)}_{\mathrm{NLL4L}} = \left\{\left\{\tfrac{1}{2}(\sigma - 1)\right\}\right\}. \tag{62}$$

Note that this lower limit also features the correct power growth, see (15), as g (see (13)) diverges, and is the best possible, being saturated by the potential (34) with (35).

Let us recall that a somewhat analogous upper limit $L^{(+)}_{\mathrm{eff}}$ on the largest value L of ℓ for which the potential $V(r)$ possesses bound states (entailing that for $\ell > L^{(+)}_{\mathrm{eff}}$ the potential $V(r)$ certainly does not possess any ℓ-wave bound state), which we denote as ULL, reads

$$\text{ULL}: \quad L \leqslant L^{(+)}_{\mathrm{eff}} = \left\{\left\{\tfrac{1}{2}(\pi\sigma - 1)\right\}\right\}. \tag{63}$$

(Indeed, it is an immediate consequence—via a standard comparison argument, see below—of the well-known fact that the solution $u(r)$ characterized by the boundary condition $u(0) = 0$ of the ODE $r^2u''(r) + cu(r) = 0$ features a zero in $0 < r < \infty$ only if the real constant c exceeds $\frac{1}{4}$, $c > \frac{1}{4}$.)

1.4. Limits defined in terms of comparison potentials

The results reported in this subsection are directly based on the elementary remark that, if $V^{(1)}(r) \leqslant V^{(2)}(r)$ for all values of $r, 0 \leqslant r < \infty$, then the number $N^{(2)}_\ell$ of ℓ-wave bound states associated with the potential $V^{(2)}(r)$ cannot exceed the number $N^{(1)}_\ell$ of ℓ-wave bound states associated with the potential $V^{(1)}(r)$, $N^{(1)}_\ell \geqslant N^{(2)}_\ell$.

Let $V(r)$ satisfy the negativity condition (2) and let $H^{(\ell)}_\lambda(r)$ be the 'additional' (ℓ-dependent) potential defined as follows (see section 3):

$$H^{(\ell)}_\lambda(r) = -\frac{\ell(\ell+1)}{r^2} + \frac{5}{16}\left(\frac{V'(r)}{V(r)}\right)^2 + \frac{V''(r)}{4|V(r)|} + (1 - 4\lambda^2)|V(r)| \tag{64}$$

with λ being an arbitrary *non-negative* constant, $\lambda \geqslant 0$. Then the following limits on the number N_ℓ of ℓ-wave bound states possessed by the potential $V(r)$ hold:

$$N_\ell \geqslant \{\{\lambda S\}\} \qquad \text{if} \quad H^{(\ell)}_\lambda(r) \geqslant 0 \quad \text{for} \quad 0 \leqslant r < \infty \tag{65}$$

$$N_\ell \leqslant \{\{\lambda S\}\} \qquad \text{if} \quad H^{(\ell)}_\lambda(r) \leqslant 0 \quad \text{for} \quad 0 \leqslant r < \infty \tag{66}$$

where S is defined by (7) and the double braces denote the integer part. Note however that, for higher partial waves ($\ell > 0$), the lower limit, (65), is applicable only to potentials that vanish at the origin ($r = 0$) at least proportionally to $r^{4\ell}$ and asymptotically ($r \to \infty$) no faster than $r^{-4(\ell+1)}$, while for S-waves ($\ell = 0$), the upper limit is only applicable to potentials that vanish asymptotically proportionally to r^{-p} with (see (3)) $2 < p \leqslant 4$.

Note in particular that (the special case with $\ell = 0$ and $\lambda = 1/2$ of) this result implies that, for any potential $V(r)$ that satisfies, in addition to the negativity condition (2), the inequality

$$\frac{5}{4}\left(\frac{V'(r)}{V(r)}\right)^2 - \frac{V''(r)}{V(r)} \geqslant 0 \qquad \text{for} \quad 0 \leqslant r < \infty \tag{67}$$

there holds the following *new* lower bound on the number N_0 of S-wave bound states:

$$N_0 \geqslant \left\{\left\{ \frac{S}{2} \right\}\right\}. \tag{68}$$

As can be easily verified, this lower limit is for instance applicable to the (class of) potential(s)

$$V(r) = -\frac{g^2}{R^2}\left(\frac{r}{R}\right)^{\alpha-2} \exp\left[-\left(\frac{r}{R}\right)^{\beta}\right] \tag{69}$$

where α and β are two arbitrary *positive* constants, $\alpha > 0$, $\beta > 0$, that satisfy the following condition:

$$\alpha\beta \geqslant \beta^2 + 1. \tag{70}$$

It then yields the explicit lower limit

$$N_0 \geqslant \left\{\left\{ \frac{g}{\pi\beta} 2^{\alpha/2\beta} \Gamma\left(\frac{\alpha}{2\beta}\right) \right\}\right\}. \tag{71}$$

In particular, when $\alpha = 2$ and $\beta = 1$, we obtain the lower limit $N_0 \geqslant \{\{2g/\pi\}\}$ on the number of S-wave bound states for the exponential potential which simplifies and improves the lower bound found in our previous work (see equation (2.13) of [4]).

The particular case of the special potential $V(r)$ that yields via (64) $H_\lambda^{(\ell)}(r) = 0$ is investigated elsewhere [5].

1.5. Limits of second kind, defined in terms of recursive formulae

In this section we exhibit *new* upper and lower limits on the number of bound states, defined in terms of recursive formulae which are particularly convenient for numerical computation. We call these limits 'of second kind', following the terminology introduced in [4]. It is possible, following [4], to derive such limits directly for the number N_ℓ of ℓ-wave bound states possessed by a central potential $V(r)$ having some monotonicity properties, via a treatment based on the ODE (191) (see section 3) and utilizing the potential (34) that, as discussed in section 3, trivializes the solution of this ODE (just as the square-well potential employed in [4] to obtain this kind of result trivializes the S-wave version of this ODE). But the results we obtained in this manner, including the precise monotonicity conditions on the potential $V(r)$ required for their validity (although, as a matter of fact, the simple monotonicity condition (1) would be more than enough for the validity of the upper limit), are not sufficiently neat, nor are they expected to be sufficiently stringent, to warrant our reporting them here. We rather focus on the derivation, using essentially the same technique employed in [4], of *new* upper and lower limits on the number N_0 of S-wave bound states possessed by a (nonmonotonic) potential $V(r)$ that has two zeros, $V(r_\pm) = 0$, and that is positive for r smaller than r_-, negative for r in the interval from r_- to r_+ with only one minimum, say at $r = r_{\min}$, in this interval, and is again positive for r larger than r$_+$, see (43). Indeed, as already noted in subsection 1.2, one can then obtain *new* upper and lower limits on the number N_ℓ of ℓ-wave bound states by replacing the potential $V(r)$ with the effective potential $V_{\ell,\mathrm{eff}}(r)$, see (24), since such a potential, for a fairly large class of potentials $V(r)$, does indeed satisfy the shape conditions mentioned above, see (50).

To get the *new* upper limit one introduces the following two recursions:

$$r_{j+1}^{(\mathrm{up,incr})} = r_j^{(\mathrm{up,incr})} + \frac{\pi}{2}\left|V\left(r_j^{(\mathrm{up,incr})}\right)\right|^{-1/2} \qquad \text{with} \quad r_0^{(\mathrm{up,incr})} = r_{\min} \tag{72}$$

$$r_{j+1}^{(\mathrm{up,decr})} = r_j^{(\mathrm{up,decr})} - \frac{\pi}{2}\left|V\left(r_j^{(\mathrm{up,decr})}\right)\right|^{-1/2} \qquad \text{with} \quad r_0^{(\mathrm{up,decr})} = r_{\min} \tag{73}$$

that define the increasing, respectively decreasing, sequences of radii $r_j^{(\mathrm{up,incr})}$, respectively $r_j^{(\mathrm{up,decr})}$, both starting from the value r_{min} at which the potential attains its minimum value, see (43). Now let $J^{(\mathrm{up,incr})}$ be the first value of j such that $r_j^{(\mathrm{up,incr})}$ exceeds or equals r_+,

$$r^{(\mathrm{up,incr})}_{J^{(\mathrm{up,incr})}-1} < r_+ \leqslant r^{(\mathrm{up,incr})}_{J^{(\mathrm{up,incr})}} \tag{74}$$

and likewise let $J^{(\mathrm{up,decr})}$ be the first value of j such that $r_j^{(\mathrm{up,decr})}$ becomes smaller than, or equal to, r_-,

$$r^{(\mathrm{up,decr})}_{J^{(\mathrm{up,decr})}} \leqslant r_- < r^{(\mathrm{up,decr})}_{J^{(\mathrm{up,decr})}-1}. \tag{75}$$

The *new* upper limit of the second kind (ULSK) is then provided by the neat formula

$$\text{ULSK:}\quad N_0 < \tfrac{1}{2}\left[J^{(\mathrm{up,incr})} + J^{(\mathrm{up,decr})} + 1 + \theta\left(r^{(\mathrm{up,decr})}_{J^{(\mathrm{up,decr})}}\right)\right]. \tag{76}$$

Here $\theta(x)$ is the standard step function, $\theta(x) = 1$ if $x \geqslant 0$, $\theta(x) = 0$ if $x < 0$, and $r^{(\mathrm{up,decr})}_{J^{(\mathrm{up,decr})}}$ is the 'last' (smallest) radius yielded by the recursion (73), see (75). (Of course the 'θ-term' on the right-hand side of formula (76) is not very significant, at least for potentials possessing many bound states, namely just when this upper limit is more likely to be quite cogent, see section 2).

To obtain a lower limit one must instead define the following increasing, respectively decreasing, sequences of radii $r_j^{(\mathrm{lo,incr})}$, respectively $r_j^{(\mathrm{lo,decr})}$:

$$r^{(\mathrm{lo,incr})}_{j+1} = r^{(\mathrm{lo,incr})}_{j} + \frac{\pi}{2}\left|V\left(r^{(\mathrm{lo,incr})}_{j}\right)\right|^{-1/2} \qquad \text{with} \quad r_0^{(\mathrm{lo,incr})} > r_- \tag{77}$$

$$r^{(\mathrm{lo,decr})}_{j+1} = r^{(\mathrm{lo,decr})}_{j} - \frac{\pi}{2}\left|V\left(r^{(\mathrm{lo,decr})}_{j}\right)\right|^{-1/2} \qquad \text{with} \quad r_0^{(\mathrm{lo,decr})} < r_+. \tag{78}$$

Now let $J^{(\mathrm{lo,incr})}$ be the first value of j such that $r_j^{(\mathrm{lo,incr})}$ exceeds or equals r_{min},

$$r^{(\mathrm{lo,incr})}_{J^{(\mathrm{lo,incr})}-1} < r_{\mathrm{min}} \leqslant r^{(\mathrm{lo,incr})}_{J^{(\mathrm{lo,incr})}} \tag{79}$$

and likewise let $J^{(\mathrm{up,decr})}$ be the first value of j such that $r_j^{(\mathrm{up,decr})}$ becomes smaller than, or equal to, r_{min},

$$r^{(\mathrm{lo,decr})}_{J^{(\mathrm{lo,decr})}} \leqslant r_{\mathrm{min}} < r^{(\mathrm{lo,decr})}_{J^{(\mathrm{lo,decr})}-1}. \tag{80}$$

The *new* lower limit of the second kind (LLSK) is then provided by the neat formula

$$\text{LLSK:}\quad N_0 > \tfrac{1}{2}\left(J^{(\mathrm{lo,incr})} + J^{(\mathrm{lo,decr})} - H\right) - 1 \tag{81}$$

where the parameter H vanishes, $H = 0$, provided either $\left|V\left(r^{(\mathrm{lo,incr})}_{J^{(\mathrm{lo,incr})}-1}\right)\right| \leqslant \left|V\left(r^{(\mathrm{lo,decr})}_{J^{(\mathrm{lo,decr})}-1}\right)\right|$ and $r^{(\mathrm{lo,incr})}_{J^{(\mathrm{lo,incr})}}$ does not exceed $r^{(\mathrm{lo,decr})}_{J^{(\mathrm{lo,decr})}-1}$, $r^{(\mathrm{lo,incr})}_{J^{(\mathrm{lo,incr})}} \leqslant r^{(\mathrm{lo,decr})}_{J^{(\mathrm{lo,decr})}-1}$, or $\left|V\left(r^{(\mathrm{lo,incr})}_{J^{(\mathrm{lo,incr})}-1}\right)\right| \geqslant \left|V\left(r^{(\mathrm{lo,decr})}_{J^{(\mathrm{lo,decr})}-1}\right)\right|$ and $r^{(\mathrm{lo,incr})}_{J^{(\mathrm{lo,incr})}-1}$ does not exceed $r^{(\mathrm{lo,decr})}_{J^{(\mathrm{lo,decr})}}$, $r^{(\mathrm{lo,incr})}_{J^{(\mathrm{lo,incr})}-1} \leqslant r^{(\mathrm{lo,decr})}_{J^{(\mathrm{lo,decr})}}$, and it is unity, $H = 1$, otherwise. (Anyway this term does not make a very significant contribution, at least for potentials possessing many bound states, when this lower limit is more likely to be quite cogent, see section 2.) Note moreover that, in the recursions (77), respectively (78), the starting points, $r_0^{(\mathrm{lo,incr})}$, respectively $r_0^{(\mathrm{lo,decr})}$, are only restricted by inequalities; of course interesting results will be obtained only by assigning $r_0^{(\mathrm{lo,incr})}$ relatively, but not exceedingly, close to r_-, and $r_0^{(\mathrm{lo,decr})}$ relatively, but not exceedingly, close to r_+ (to get some understanding of which choices of these parameters, $r_0^{(\mathrm{lo,incr})}$ and $r_0^{(\mathrm{lo,decr})}$, are likely to produce more cogent results, the interested reader is referred to the proof of the lower limit given in section 3; of course numerically one can make a search for the values of these parameters, $r_0^{(\mathrm{lo,incr})}$, respectively $r_0^{(\mathrm{lo,decr})}$, that maximize the right-hand side of (81), starting from values close to r_-, respectively r_+).

1.6. Limits on the total number of bound states

Clearly if $N_\ell^{(-)}$, respectively $N_\ell^{(+)}$, provides lower, respectively upper, limits on the number N_ℓ of ℓ-wave bound states, and likewise $L^{(-)}$, respectively $L^{(+)}$, provides lower, respectively upper, limits on the largest value L of the angular momentum quantum number ℓ for which the potential $V(r)$ does possess bound states, it is plain that the quantities

$$N^{(\pm)} = N(L^{(\pm)}) \tag{82a}$$

where

$$N(L) = \sum_{\ell=0}^{L} (2\ell + 1) N_\ell^{(\pm)} \tag{82b}$$

provide lower, respectively upper, limits,

$$N^{(-)} \leqslant N \leqslant N^{(+)} \tag{83}$$

to the total number N, see (6), of bound states possessed by the potential $V(r)$. Hence several such limits can be easily obtained from the results reported above.

Remark. There is however a significant loss of accuracy in using these formulae, (82) and (83), to obtain upper or lower limits on the total number of bound states N. Note that the upper limit, L, of the sum on the right-hand side of (82*b*) must be an integer, but after the sum has been performed to calculate $N(L)$, see (82*b*), it gets generally replaced by a noninteger number, $L^{(+)}$, respectively $L^{(-)}$, to evaluate the upper, respectively lower, limit $N^{(+)}$, respectively $N^{(-)}$, see (82*a*) and (83). Let us illustrate this effect by a fictitious numerical example. Suppose we were able to prove, say, the lower limit $N_\ell > \frac{13}{3} - \ell$. We would then know that $N_0 > \frac{13}{3}, N_1 > \frac{10}{3}, N_2 > \frac{7}{3}, N_3 > \frac{4}{3}, N_4 > \frac{1}{3}$ entailing $N_0 \geqslant 5, N_1 \geqslant 4, N_2 \geqslant 3, N_3 \geqslant 2, N_4 \geqslant 1$, hence we could conclude that there are at least 55 bound states $(5 + 12 + 15 + 14 + 9 = 55)$, $N \geqslant 55$. But by the above procedure we would infer that $L^{(-)} = \frac{10}{3}$ and $N(L) = \frac{1}{6}(L + 1)(26 + 21L - 4L^2)$ entailing $N^{(-)} > 37.23$ hence $N \geqslant 38$. This is a much less stringent (lower) limit. Clearly, due to the round-off errors, a lot of information got lost. This defect can be remedied, but only marginally, by inserting in the expression $N(L)$ the best value of $L^{(-)}$ yielded by the above fictitious lower bound, namely $L^{(-)} = 4$, since we then obtain $N^{(-)} > 38.3$ hence $N \geqslant 39$. And the analogous calculation via (82) and (83) from a hypothetical upper limit $N_\ell < \frac{13}{3} - \ell$, which clearly entails $N_0 \leqslant 4, N_1 \leqslant 3, N_2 \leqslant 2, N_3 \leqslant 1$ and hence $N \leqslant 30$, yields again a less stringent result, namely the upper limit $N \leqslant 37$ if $L^{(+)} = \frac{10}{3}$ is used. This limit can be slightly improved, namely $N \leqslant 35$, if the integer part of $L^{(+)} = 3$ is used. In the following we have tried to take care of this problem to the extent possible compatible with the goal of obtaining simple explicit formulae.

In the next section we illustrate the remark just made by computing firstly, via the upper and lower limits NUL2ℓ and NLL2ℓ, two sets of integers $N_\ell^{(-)}$ and $N_\ell^{(+)}$ such that $N_\ell^{(-)} \leqslant N_\ell \leqslant N_\ell^{(+)}$, and then by evaluating upper and lower limits $N^{(\pm)}$ on the total number N of bound states, see (83), via the standard formula (82*b*) with L replaced, as it were, by ∞, the sum being automatically stopped by the vanishing of the summand. The upper and lower limits obtained with this procedure will be called NUL2N and NLL2N, respectively.

Anyway in this subsection some results obtained via (82) and (83) are reported, namely those which we believe deserve to be displayed thanks to their neat character. But firstly let us briefly review the upper limits on the total number of bound states N previously known (we did not find any lower limits on N in the literature).

A classical result, the validity of which is not restricted to central potentials, is known in the literature as the Birman–Schwinger upper bound [2, 22], and we denote it as BiS. It reads as follows:

$$\text{BiS}: \quad N < \frac{1}{(4\pi)^2} \int \mathrm{d}^3\vec{r}_1 \, \mathrm{d}^3\vec{r}_2 \frac{V^{(-)}(\vec{r}_1) V^{(-)}(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^2} \tag{84}$$

implying, for central potentials,

$$\text{BiScentral}: \quad N < \frac{1}{2} \int_0^\infty \mathrm{d}r_1 \, r_1 V^{(-)}(r_1) \int_0^\infty \mathrm{d}r_2 \, r_2 V^{(-)}(r_2) \log\left|\frac{r_1 + r_2}{r_1 - r_2}\right|. \tag{85}$$

This upper limit, however, is proportional to g^4 (see (13)) rather than g^3 (see (16)), hence it provides a limit much larger than the exact result for strong potentials possessing many bound states.

A simple upper limit, that we denote by BSN, can be obtained from the BSℓ upper limit, see (17); it reads

$$\text{BSN}: \quad N < \{\{I\}\} \left\{\left\{\frac{I+1}{2}\right\}\right\} \tag{86}$$

with

$$I = \int_0^\infty \mathrm{d}r \, r[-V^{(-)}(r)]. \tag{87}$$

This upper limit is also proportional to g^4 rather than g^3.

An upper limit that does not have this defect and is also valid for potentials that need not be central was obtained by Lieb [18]. We denote it as L:

$$\text{L}: \quad N < 0.116 \int \mathrm{d}^3\vec{r}[-V^{(-)}(\vec{r})]^{3/2} \tag{88}$$

(for the origin of the numerical coefficient on the right-hand side of this formula, we refer to the original paper [18]). For central potentials it reads as follows:

$$\text{Lcentral}: \quad N < 1.458 \int_0^\infty \mathrm{d}r \, r^2[-V^{(-)}(r)]^{3/2} \tag{89}$$

(the numerical coefficient in this formula is obtained by multiplying that in the preceding formula by 4π; for other results of this kind, none of which seems however to be more stringent than those reported here, see [3]).

Let us conclude this listing of previously *known* results by reporting the upper limit on the total number of bound states N obtained [11] by inserting (21) and (22) in (82). As entailed by its origin, it only holds for monotonically nondecreasing potentials, see (1) and (2). We denote it as CMSN:

$$\text{CMSN}: \quad N < \frac{\pi^2}{12}\left[S^3 + 3S^2 + \frac{2}{\pi}\left(3 - \frac{1}{2\pi}\right)S + \frac{3}{\pi^2}\right] \tag{90}$$

with S defined by (7).

We did not obtain any *new* upper limit on the total number N of bound states sufficiently neat to be worth reporting. We report instead a rather trivial upper limit on N obtained via (82) with L replaced by its upper limit $L_{\text{eff}}^{(+)}$ (see (63)) and with $N_\ell \leqslant N_0$ and N_0 bounded above by NUL2, see (44). This upper limit on the total number N of bound states is therefore applicable to potentials that satisfy condition (36), and we denote it as NUL2Nn:

$$\text{NUL2Nn}: \quad N < \frac{1}{8}(\pi\sigma + 1)^2 \left\{2 + S + \frac{1}{\pi}\log\left[\frac{-V^{(-)}(r_{\min})}{M}\right]\right\} \tag{91}$$

with M defined by (46), σ defined by (8), S defined by (7) and $V^{(-)}(r_{\min})$ the minimal value of (the negative part of) the potential, see (36). For a *monotonic* potential, see (1), this upper limit takes the simpler form

$$\text{NUL2Nm}: \quad N < \frac{1}{8}(\pi\sigma + 1)^2 \left\{1 + S + \frac{1}{2\pi} \log\left[\frac{V(p)}{V(q)}\right]\right\} \tag{92}$$

where p and q are defined by (47) and (48), respectively (this result is obtained using, instead of NUL2, the analogous result valid for monotonic potentials [4]). It is remarkable that, in spite of the drastic approximation $N_\ell \leqslant N_0$ used to obtain these two limits, they turn out, in all the tests performed in section 2, to be more stringent than all previously known results.

We now report two *new* lower limits, which recommend themselves because of their neatness, although, for the reason outlined above, one cannot expect them to be very stringent.

A *new* lower limit, that we denote as NLLN3, on the total number N of bound states for a potential $V(r)$ that satisfies conditions (36), follows from the lower limits NLL3, see (59), and NLL3L, see (60). A simple calculation yields

$$\text{NLLN3}: \quad N > \frac{\nu}{6\lambda^2}(2\nu + \lambda)(\nu + \lambda) \tag{93}$$

with ν defined by (59*b*) and

$$\lambda = \frac{1}{\pi} \log\left(\frac{q}{p}\right) \tag{94}$$

with q and p defined by (47) and (48) (we assume $q \geqslant p$, hence $\lambda \geqslant 0$).

Another *new* lower limit, that we denote as NLLN4, on the total number N of bound states for a potential $V(r)$ that satisfies condition (5) is implied, via (82), from the lower limits NLL4, see (61), and NLL4L, see (62). A simple calculation yields

$$\text{NLLN4}: \quad N \geqslant \frac{1}{2}\left\{\left\{\left(\frac{\sigma+1}{2}\right)\right\}\right\}\left\{\left\{\left(\frac{\sigma+3}{2}\right)\right\}\right\} \tag{95}$$

with σ defined by (8). Here, as usual, the double braces denote the integer part. This lower limit has the merit of being rather neat, but it increases proportionally to g^2 (see (13)) rather than g^3 (see (16)), hence it cannot be expected to be cogent for strong potentials possessing many bound states.

2. Tests

In this section we test the efficiency of the *new* limits reported in section 1 by comparing them for some representative potentials with the exact results and with the results obtained via previously *known* limits. For these tests we use three different potentials: the Morse potential [21] (hereafter referred to as M)

$$\text{M}: \quad V(r) = -g^2 R^{-2}\left\{2\exp\left[-\left(\frac{r}{R} - \alpha\right)\right] - \exp\left[-2\left(\frac{r}{R} - \alpha\right)\right]\right\} \tag{96}$$

the exponential potential (hereafter referred to as E)

$$\text{E}: \quad V(r) = -g^2 R^{-2} \exp\left(-\frac{r}{R}\right) \tag{97}$$

and the Yukawa potential (hereafter referred to as Y)

$$\text{Y}: \quad V(r) = -g^2 (rR)^{-1} \exp\left(-\frac{r}{R}\right). \tag{98}$$

In all these equations, and below, R is an arbitrary (positive) given radius, and g, as well as α in (96), are arbitrary dimensionless *positive* constants.

2.1. Tests of the limits on the number of bound states N_ℓ

The first potential we use to test the new limits is the M potential (96). This is a nonmonotonic potential for which the number N_0 of bound states for vanishing angular momentum is known; we indeed consider for this potential only the $\ell = 0$ case. (We do not test the GGMT and CMS2 limits with this M potential since, from their incorrect behaviour when the strength g of the potential diverges, we already know that these limits give poor results. But, in spite of this incorrect behaviour, these limits could be useful when there are few bound states; they are therefore tested below with the E and Y potentials, in the cases with $\ell > 0$).

The exact formula for the number of S-wave bound states for the M potential is

$$N_0 = \left\{\left\{g + \frac{1}{2}\right\}\right\}. \tag{99}$$

Note that it is independent of the value of the constant α.

For this potential, the limits NUL2 and NLL2, see (44) and (45), can be computed (almost completely) analytically:

$$\text{NUL2}: \quad N_0 < g - \frac{1}{2\pi}\log s + 1 \tag{100}$$

$$\text{NLL2}: \quad N_0 > g + \frac{1}{2\pi}\log s - \frac{3}{2} \tag{101}$$

with $s = \min(2y - y^2, 2x - x^2)$ and x, y solutions of

$$\pi - \sqrt{2y - y^2} - 2\arcsin\left(\frac{y}{2}\right) = \frac{\pi}{2g} \tag{102}$$

$$\sqrt{2x - x^2} + 2\arcsin\left(\frac{x}{2}\right) = \frac{\pi}{2g}. \tag{103}$$

The calculation of the cut-off radii p and q, see (47) and (48), cannot be evaluated analytically. But one can compute upper and lower limits, $\tilde{p} < p$ and $\tilde{q} > q$, on these radii by using only the attractive part of the potential in the definition (47) and (48) of p and q. When $\tilde{p}$ and $\tilde{q}$ are used in the place of p and q we obtain the (marginally less stringent) limits (denoted as NUL2s and NLL2s)

$$\text{NUL2s}: \quad N_0 < g + \frac{1}{2\pi}\log\left[\frac{z^4}{4(z^2-1)}\right] + 1 \tag{104}$$

$$\text{NLL2s}: \quad N_0 > g - \frac{1}{2\pi}\log\left[\frac{z^4}{4(z^2-1)}\right] - \frac{3}{2} \tag{105}$$

with $z = 8g/\pi$. As mentioned in section 1, validity of the inequalities $\tilde{p} \leqslant r_{\min} \leqslant \tilde{q}$ is required in order to use the NUL2s and NLL2s limits; this leads to the restriction $g \geqslant \pi\sqrt{2}/(8(\sqrt{2}-1)) \cong 1.34$.

The NLL1 limit (38) takes for the M potential the simple form

$$\text{NLL1}: \quad N_0 > 0.672g - 1. \tag{106}$$

The other *new* limits cannot be tested with this potential: the upper limit NUL1 and the limits of the second kind are not applicable because $r_+ = \infty$, the lower limit NLL4 is only applicable to monotonic potentials, and the lower limit NLL3 coincides with NLL2 for $\ell = 0$.

The previously known limits applicable to this potential (note that the CMS upper limit is only applicable to monotonic potentials) take the form

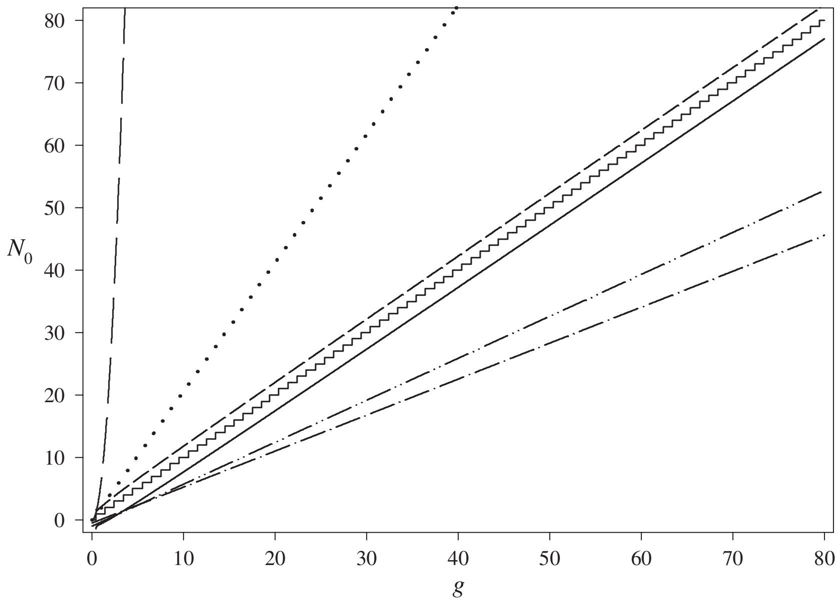

Figure 1. Comparison between the exact value (99) of N_0 (ladder), the upper limits BSℓ (107) (long-dashed), Mℓ (108) (dotted) and NUL2s (104) (short-dashed) and the lower limits Cℓ (109) (dashed-dotted), NLL2s (105) (solid) and NLL1 (106) (dashed-double-dotted) for the M potential (96) (all with $\ell = 0$ and $\alpha = 1$).

$$\mathrm{BS}\ell: \quad N_0 \leqslant 2g^2\left(\alpha + \log 2 + \tfrac{3}{2}\right) \tag{107}$$

$$\mathrm{M}\ell: \quad N_0 \leqslant \sqrt{2}g(\alpha^2 + (3 - 2\log 2)\alpha + 1.901)^{1/4} \tag{108}$$

$$\mathrm{C}\ell: \quad N_0 > -\frac{1}{2} + \frac{2}{\pi} g T(\alpha) \tag{109a}$$

$$T(\alpha) = \max_{0 \leqslant \gamma \leqslant 1} \left\{ \frac{1}{\gamma} \left[\exp(\alpha) - \frac{1}{4}\exp(2\alpha) - \sqrt{1-\gamma^2} + \frac{\gamma^2}{2} \log\left(\frac{1+\sqrt{1-\gamma^2}}{1-\sqrt{1-\gamma^2}} \right) \right] \right\}. \tag{109b}$$

To obtain this last limit we set $a = R/(\gamma g)$ in (30) with $\ell = 0$. Note that for $\alpha > \alpha_0 \cong 1.386$, with $4\,\exp(\alpha_0) = \exp(2\alpha_0)$, the lower limit C$\ell$ is trivial because $T(\alpha)$ is then negative. Note that, for $\alpha = \log 2$, $T(\alpha)$ reaches the maximal value: $T(\alpha) = 1.055$. The factor multiplying g on the right-hand side of (109a) coincides with that multiplying g on the right-hand side of the NLL1 lower limit (106); so for this particular value of α, this Cℓ limit is slightly more stringent (of course for all values of g) than the NLL1 lower limit. For all other values of α, there exists a value g_α such that for all $g \geqslant g_\alpha$, the NLL1 limit is more stringent than the Cℓ limit. For example, for $\alpha = 1$, the NLL1 limit yields more cogent results than the Cℓ limit when $g \geqslant 5.22$, namely when the number of bound states is greater than 5. Figure 1 shows these limits as a function of g. Note that some limits depend on α while the exact result does not. We tested the results for the $\alpha = 1$ case (not $\alpha = 0$, in order to have a nonmonotonic potential). It is clear from this figure that the generalizations to nonmonotonic potentials of the results obtained in [4], namely the limits NUL2 and NLL2, are quite cogent. This remains true even for large values of g: for instance, when the exact number N_0 of bound states is 5000, these upper and lower limits restrict its value to the rather small interval [4996, 5003]. In this case the BSℓ upper limit exceeds 1.5×10^8, the Mℓ upper limit only reveals that $N < 10\,307$, the lower limit Cℓ that $N > 2879$ and the lower limit NLL1 that $N > 3359$.

The second test is performed with the E potential (97). The exact number N_ℓ of bound states for this potential is computed by integrating numerically (191) with $\eta(0) = 0$ and $\eta(\infty) = N_\ell \pi$, see section 3.

The upper limit NUL1ℓ reads

$$\text{NUL1}\ell\colon \quad N_\ell < 1 + \frac{2}{\pi}\sqrt{x_+ - x_-}\left\{g^2[\exp(-x_-) - \exp(-x_+)] - \ell(\ell+1)\left(\frac{1}{x_-} - \frac{1}{x_+}\right)\right\}^{1/2} \tag{110}$$

where $x_\pm$ are the two solutions of

$$\ell(\ell+1) = g^2 x_\pm^2 \exp(-x_\pm). \tag{111}$$

The NLL1nℓ limit reads

$$\text{NUL1n}\ell\colon \quad N_\ell > -1 + \frac{1}{\pi}\left\{g^2[\exp(-x_-) - \exp(-x_+)] - \ell(\ell+1)\left(\frac{1}{x_-} - \frac{1}{x_+}\right)\right\}\left|V_{\ell,\text{eff}}^{\min}\right|^{-1/2} \tag{112}$$

where $V_{\ell,\text{eff}}^{\min}$ is the minimal value of the effective potential (24). The NUL2ℓ and NLL2ℓ limits can be written as follows:

$$\text{NUL2}\ell\colon \quad N_\ell < \frac{1}{\pi}F(g, \ell; x_-, x_+) + \frac{1}{2\pi}\log\left|\frac{V_{\ell,\text{eff}}^{\min}}{M}\right| + 1 \tag{113}$$

$$\text{NLL2}\ell\colon \quad N_\ell > \frac{1}{\pi}F(g, \ell; x_-, x_+) - \frac{1}{2\pi}\log\left|\frac{V_{\ell,\text{eff}}^{\min}}{M}\right| - \frac{3}{2} \tag{114}$$

where

$$F(g, \ell; a, b) = \int_a^b \frac{\mathrm{d}x}{x}\sqrt{g^2x^2\exp(-x) - \ell(\ell+1)} \tag{115}$$

and

$$M = \min[|V_{\ell,\text{eff}}(p)|, |V_{\ell,\text{eff}}(q)|] \tag{116a}$$

where p and q are solutions of

$$F\left(g, \ell; x_-, \frac{p}{R}\right) = \frac{\pi}{2} \qquad F\left(g, \ell; \frac{q}{R}, x_+\right) = \frac{\pi}{2}. \tag{116b}$$

The lower limits NLL3 and NLL4 take much simpler forms

$$\text{NLL3}\colon \quad N_\ell > \frac{2}{\pi}g - \frac{1}{2\pi}\log\left(\frac{4g}{\pi}\right) - \frac{3}{2} - \frac{\ell}{\pi}\log\left[\frac{\log x}{\log(1-x)}\right] \tag{117}$$

with $x = \pi/(4g)$, and

$$\text{NLL4}\colon \quad N_\ell > \frac{2g}{\pi e(2\ell+1)} - \frac{1}{2}. \tag{118}$$

The previously known limits are found to be

$$\text{BS}\ell\colon \quad N_\ell < \frac{g^2}{2\ell+1} \tag{119}$$

Table 1. Comparison for the E potential (97) between the exact number N_ℓ of bound states, various upper and lower limits on N_ℓ previously known and new upper and lower limits on N_ℓ, for several values of g and ℓ.

g	ℓ	LLSK	NLL3	NLL1nℓ	NLL2ℓ	Ex.	NUL2ℓ	BSℓ	CMS	Mℓ	GGMT	NUL1ℓ	ULSK
8	1	3	3	3	3	**4**	5	21	9	8	21	12	6
	3	1	1	2	1	**2**	3	9	8	5	6	6	4
13	2	5	4	4	5	**7**	8	33	15	13	31	18	9
	6	2	0	2	2	**3**	4	13	13	7	7	7	5
18	3	7	6	6	7	**9**	10	46	21	18	43	25	11
	9	2	0	3	2	**4**	4	17	17	9	8	9	5
24	4	10	8	8	10	**12**	13	64	28	24	60	33	14
	12	4	0	4	4	**5**	6	23	23	13	11	12	7
29	5	13	9	9	13	**15**	16	76	34	29	71	40	17
	15	4	0	5	4	**6**	7	27	28	15	13	13	8
35	6	16	11	11	16	**18**	19	94	41	35	88	49	20
	18	6	0	6	6	**7**	8	33	33	18	16	16	9
40	7	18	12	13	18	**20**	21	106	47	40	100	55	22
	21	6	0	6	6	**8**	9	37	38	20	18	18	10

$$\text{CMS}: \quad N_\ell < \frac{4g}{\pi} + 1 - \sqrt{1 + \frac{4\ell(\ell+1)}{\pi^2}} \tag{120}$$

$$\text{M}\ell: \quad N_\ell < (AB)^{1/4} \tag{121a}$$

where A and B are given by

$$A = g^2\left[\exp(-x_-)(x_-^2 + 2x_- + 2) - \exp(-x_+)\left(x_+^2 + 2x_+ + 2\right)\right] - \ell(\ell+1)(x_+ - x_-) \tag{121b}$$

$$B = g^2[\exp(-x_-) - \exp(-x_+)] - \ell(\ell+1)\left(\frac{1}{x_-} - \frac{1}{x_+}\right) \tag{121c}$$

$$\text{GGMT}: \quad N_\ell \leqslant g^{2p}(2\ell+1)^{(1-2p)}\frac{C_p\Gamma(2p)}{p^{2p}} \tag{122}$$

with C_p defined by (25b)

$$\text{CMS2}: \quad N_\ell < g^{2p}(2\ell+1)^{(1-2p)}\frac{\tilde{C}_p\Gamma(2p)}{p^{2p}} \tag{123}$$

with $\tilde{C}_p$ defined by (26b)

$$\text{C}\ell: \quad N_\ell > \frac{2g}{\pi(2\ell+1)} y \exp\left(-\frac{y}{2}\right) - \frac{1}{2} \tag{124}$$

where y is the solution of $y\exp(-y) = (2\ell+1)y^{2\ell}\Gamma(1-2\ell, y)$.

Comparisons between the various limits and the exact results are presented in table 1. The BSℓ limit gives poor results when g becomes large but becomes slightly better as ℓ increases. The CMS gives better restrictions when ℓ is small but behaves like the BSℓ limit when ℓ increases. The Mℓ limit overestimates the number of bound states by a factor of 2 when ℓ is small; it is no better for larger ℓ, yet better than the BSℓ and CMS limits. The GGMT limit (with, in each case, the optimized value of the parameter p, see (122)) gives similar results to those yielded by the BSℓ limit when ℓ is small and becomes better and equivalent to the Mℓ

limit for larger values of ℓ. The results obtained with the CMS2 limit are uninteresting and hence not reported: indeed, the values of p which minimize the value of the limit are either $p = 1/2$ for small values of ℓ (in which case this limit is analogous but less stringent than the CC limit, see (29)), or $p = 1$ for larger values of ℓ (and this yields the BSℓ limit). The new limits NUL2ℓ and NLL2ℓ clearly yield the most stringent results. The NLL1nℓ lower limit only yields cogent results for large values of the angular momentum. The NLL3 lower limit works reasonably well for small values of ℓ but becomes poor for higher values of the angular momentum. The limits of the second kind ULSK and LLSK yield similar results to those given by the NUL2ℓ and NLL2ℓ limits. Note that the arbitrary radii $r_0^{(\mathrm{lo,incr})}$ and $r_0^{(\mathrm{lo,decr})}$ have been chosen to optimize the restriction on the number of ℓ-wave bound states. Finally, the results obtained with the Cℓn and the NLL4 lower limits are not reported because they are very poor. These limits give $N_\ell \geqslant 1$ for small values of ℓ and $N_\ell \geqslant 0$ for large values of ℓ. This defect comes from the presence of the factor $1/(2\ell + 1)$ which for instance implies that this lower bound becomes three times smaller when ℓ goes from 0 to 1 while the actual number of bound states N_ℓ decreases generally only by one or two units.

The last test is performed with the Y potential (98). The exact number N_ℓ of bound states is again computed by integrating numerically (191) with $\eta(0) = 0$ and $\eta(\infty) = N_\ell \pi$, see section 3.

The NUL1ℓ limit takes the form

$$\text{NUL1}\ell: \quad N_\ell < 1 + \frac{2}{\pi}\sqrt{x_+ - x_-}\left\{g^2\int_{x_-}^{x_+} \mathrm{d}x\,\frac{\exp(-x)}{x} - \ell(\ell+1)\left(\frac{1}{x_-} - \frac{1}{x_+}\right)\right\}^{1/2} \tag{125}$$

where $x_\pm$ are the two solutions of the following equation

$$\ell(\ell+1) = g^2 x_\pm \exp(-x_\pm). \tag{126}$$

The NLL1nℓ limit reads

$$\text{NLL1n}\ell: \quad N_\ell > -1 + \frac{1}{\pi}\left\{g^2\int_{x_-}^{x_+} \mathrm{d}x\,\frac{\exp(-x)}{x} - \ell(\ell+1)\left(\frac{1}{x_-} - \frac{1}{x_+}\right)\right\}\left|V_{\ell,\mathrm{eff}}^{\min}\right|^{-1/2} \tag{127}$$

where $V_{\ell,\mathrm{eff}}^{\min}$ is the minimum value of the effective potential (24). The NUL2ℓ and NLL2ℓ limits can be written as follows:

$$\text{NUL2}\ell: \quad N < \frac{1}{\pi}G(g,\ell;x_-,x_+) + \frac{1}{2\pi}\log\left|\frac{V_{\ell,\mathrm{eff}}^{\min}}{M}\right| + 1 \tag{128}$$

$$\text{NLL2}\ell: \quad N > \frac{1}{\pi}G(g,\ell;x_-,x_+) - \frac{1}{2\pi}\log\left|\frac{V_{\ell,\mathrm{eff}}^{\min}}{M}\right| - \frac{3}{2} \tag{129}$$

where

$$G(g,\ell;a,b) = \int_a^b \frac{\mathrm{d}x}{x}\sqrt{g^2 x\exp(-x) - \ell(\ell+1)}. \tag{130}$$

and

$$M = \min[|V_{\ell,\mathrm{eff}}(p)|, |V_{\ell,\mathrm{eff}}(q)|] \tag{131a}$$

where p and q are solutions of

$$G\left(g,\ell;x_-,\frac{p}{R}\right) = \frac{\pi}{2} \qquad G\left(g,\ell;\frac{q}{R},x_+\right) = \frac{\pi}{2}. \tag{131b}$$

The lower limits NLL3 and NLL4 take somewhat simpler forms

$$\text{NLL3}: \quad N_\ell > \sqrt{\frac{2}{\pi}}g - \frac{x^2 - y^2}{2\pi}\log\left(\frac{4g}{\pi}\right) - \frac{3}{2} - \frac{(1+4\ell)}{2\pi}\log\left(\frac{x}{y}\right) \tag{132}$$

Table 2. Comparison for the Y potential (98) between the exact number N_ℓ of bound states, various upper and lower limits on N_ℓ previously known and new upper and lower limits on N_ℓ, for several values of g and ℓ.

g	ℓ	LLSK	NLL3	NLL1nℓ	NLL2ℓ	Ex.	NUL2ℓ	BSℓ	CMS	Mℓ	GGMT	NUL1ℓ	ULSK
8	1	3	3	2	3	**5**	6	21	12	8	19	16	7
	3	1	0	2	1	**2**	3	9	11	5	4	5	5
15	2	7	5	4	7	**9**	11	45	23	17	41	32	13
	6	2	0	3	2	**4**	5	17	20	8	8	9	8
22	3	11	8	5	11	**13**	15	69	33	25	64	48	17
	9	4	0	4	4	**5**	6	25	29	12	12	13	11
29	4	16	10	7	16	**18**	19	93	44	33	87	64	22
	12	5	0	5	6	**7**	8	33	39	16	16	18	14
35	5	19	11	8	19	**21**	23	111	53	40	104	76	26
	15	6	0	6	6	**8**	9	39	46	18	18	20	16
41	6	23	12	10	23	**25**	27	129	62	47	120	89	30
	18	7	0	6	7	**9**	10	45	54	20	20	22	18

where y and x are defined by $\mathrm{erf}(y) = \sqrt{\pi/8}/g$, $\mathrm{erf}(x) = 1 - \sqrt{\pi/8}/g$, and

$$\text{NLL4:} \quad N_\ell > \frac{g}{\pi\sqrt{e}(2\ell+1)} - \frac{1}{2}. \tag{133}$$

The previously known limits take the form

$$\text{BS}\ell\text{:} \quad N_\ell < \frac{g^2}{2\ell+1} \tag{134}$$

$$\text{CMS:} \quad N_\ell < 2\sqrt{\frac{2}{\pi}g + 1 - \sqrt{1 + \frac{4\ell(\ell+1)}{\pi^2}}} \tag{135}$$

$$\text{M}\ell\text{:} \quad N_\ell < (AB)^{1/4} \tag{136a}$$

where A and B are given by

$$A = g^2(\exp(-x_-)(1+x_-) - \exp(-x_+)(1+x_+)) - \ell(\ell+1)(x_+ - x_-) \tag{136b}$$

$$B = g^2\int_{x_-}^{x_+} \mathrm{d}x \frac{\exp(-x)}{x} - \ell(\ell+1)\left(\frac{1}{x_-} - \frac{1}{x_+}\right) \tag{136c}$$

$$\text{GGMT:} \quad N_\ell \leqslant g^{2p}(2\ell+1)^{(1-2p)} \frac{C_p\Gamma(p)}{p^p} \tag{137}$$

$$\text{CMS2:} \quad N_\ell \leqslant g^{2p}(2\ell+1)^{(1-2p)} \frac{\tilde{C}_p\Gamma(p)}{p^p} \tag{138}$$

$$\text{C}\ell\text{:} \quad N_\ell \geqslant \frac{2g}{\pi(2\ell+1)}\sqrt{y}\exp\left(-\frac{y}{2}\right) - \frac{1}{2} \tag{139}$$

where y is the solution of $\exp(-y) = (2\ell+1)y^{2\ell}\Gamma(-2\ell, y)$.

Comparisons between the various limits and the exact results are presented in table 2. The characteristics of the various limits for the Y potential are analogous to those commented on above for the E potential. Here again the NUL2ℓ and the NLL2ℓ limits are the most effective ones, being indeed fairly stringent for all the values of g and ℓ considered, and the limits of the second kind ULSK and LLSK also give quite stringent limits.

2.2. Tests of the limits for the value of L

In this subsection we test various limits on the largest value L of the angular momentum quantum number ℓ for which the potentials E and Y do possess bound states. In this paper we only obtained *new* lower limits on L. Indeed neat upper limits on L cannot be extracted from the *new* upper limits on N_ℓ presented in section 1. But we will now see that the 'naive' upper limit $L_{\mathrm{eff}}^{(+)}$ (63) is quite good and indeed better than the previously *known* upper limits $L_{\mathrm{BSL}}^{(+)}$ and $L_{\mathrm{CMSL}}^{(+)}$, see (18) and (22).

The first test is performed with the nonsingular E potential (97). The lower limit NLL3L gives

$$L_{\mathrm{NLL3L}}^{(-)} = \left\{\left\{\frac{\nu}{\lambda}\right\}\right\} \tag{140a}$$

with

$$\nu = \frac{2g}{\pi} - \frac{1}{2\pi}\log\left[\frac{1-x}{x}\right] - \frac{3}{2} \tag{140b}$$

$$\lambda = \frac{1}{\pi}\log\left[\frac{\log x}{\log(1-x)}\right] \tag{140c}$$

and $x = \pi/(4g)$. The lower limit NLL4L gives

$$L_{\mathrm{NLL4L}}^{(-)} = \left\{\left\{\frac{1}{2}\left(\frac{4g}{\pi e} - 1\right)\right\}\right\}. \tag{141}$$

The previously known upper limits on L can also be obtained analytically:

$$L_{\mathrm{BSL}}^{(+)} = \left\{\left\{\tfrac{1}{2}\left(g^2 - 1\right)\right\}\right\} \tag{142}$$

$$L_{\mathrm{CMSL}}^{(+)} = \left\{\left\{\tfrac{1}{2}\left(4g - 1\right)\right\}\right\} \tag{143}$$

$$L_{\mathrm{eff}}^{(+)} = \left\{\left\{\frac{1}{2}\left(\frac{4g}{e} - 1\right)\right\}\right\}. \tag{144}$$

A comparison between these limits and the exact results (computed numerically, as indicated in section 2.1) is presented in figure 2. Except for the BSL upper limit, we have the correct linear behaviour in g as discussed above, the best result being clearly provided by the naive upper bound $L_{\mathrm{eff}}^{(+)}$. Indeed this result appears hardly improvable, because the error introduced by this upper limit $L_{\mathrm{eff}}^{(+)}$ is of at most one unit (at least for this example, as well as the following one, see below).

The second test is performed with the singular Y potential (again the exact result can only be computed numerically). The lower limit NLL3L gives

$$L_{\mathrm{NLL3L}}^{(-)} = \left\{\left\{\frac{\nu}{\lambda}\right\}\right\} \tag{145a}$$

with

$$\nu = \sqrt{\frac{2}{\pi}}g - \frac{x^2 - y^2}{2\pi} - \frac{1}{2\pi}\log\left[\frac{x}{y}\right] - \frac{3}{2} \tag{145b}$$

$$\lambda = \frac{2}{\pi}\log\left[\frac{x}{y}\right] \tag{145c}$$

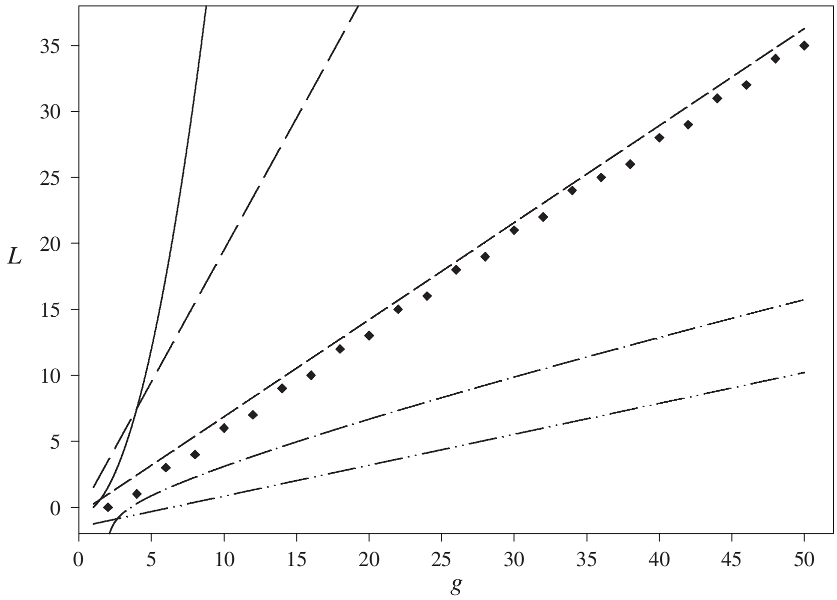

Figure 2. Comparison between the exact value of L (diamond), the upper limits BSL (142) (solid), CMSL (143) (long-dashed) and $L_{\text{eff}}^{(+)}$ (144) (short-dashed) and the lower limits NLL3L (140) (dashed-dotted) and NLL4L (141) (dashed-double-dotted) for the E potential (97).

where y and x are defined by $\text{erf}(y) = \alpha, \text{erf}(x) = 1 - \alpha, \alpha = \sqrt{(\pi/8)}/g$. The lower limit NLL4L gives

$$L_{\text{NLL4L}}^{(-)} = \left\{\left\{-\frac{1}{2} + \frac{g}{\pi\sqrt{e}}\right\}\right\}. \tag{146}$$

The previously known limits on L can also be obtained analytically:

$$L_{\text{BSL}}^{(+)} = \left\{\left\{\frac{1}{2}(g^2 - 1)\right\}\right\} \tag{147}$$

$$L_{\text{CMSL}}^{(+)} = \left\{\left\{\frac{1}{2}\left(4\sqrt{\frac{2}{\pi}}g - 1\right)\right\}\right\} \tag{148}$$

$$L_{\text{eff}}^{(+)} = \left\{\left\{\frac{1}{2}\left(\frac{2g}{\sqrt{e}} - 1\right)\right\}\right\}. \tag{149}$$

The comparison between these limits and the exact results is presented in figure 3. Here again, the best result is obtained from the naive upper bound $L_{\text{eff}}^{(+)}$ for which the error is, as indicated above, of at most one unit.

2.3. *Tests of the limits for the total number of bound states N*

This subsection is devoted to testing the limits on the total number of bound states N. We will not test the BiS limit due to its bad behaviour at large g. We do however test the BSN limit which yields the same incorrect behaviour but is simpler to compute.

The first test is performed with the E potential (97). Again, the exact result can only be calculated numerically.

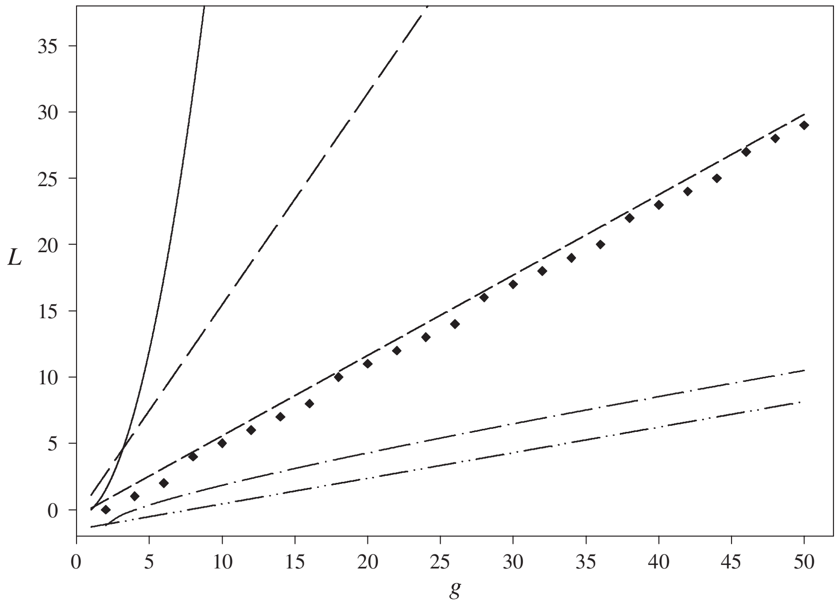

Figure 3. Comparison between the exact value of L (diamond), the upper limits BSL (147) (solid), CMSL (148) (long-dashed) and $L_{\rm eff}^{(+)}$ (149) (short-dashed) and the lower limits NLL3L (145) (dashed-dotted) and NLL4L (146) (dashed-double-dotted) for the Y potential (98).

The *new* upper limit NUL2Nm, see (92), takes the simple form

$$N < \frac{1}{8}\left(\frac{4g}{e}+1\right)^2\left(\frac{4g}{\pi}+\frac{1}{\pi}\log\frac{4g}{\pi}+1\right). \tag{150}$$

The *new* lower limit NLLN3 reads

$$N > \frac{\nu}{6}\left(2\left(L_{\rm NLL3L}^{(-)}\right)^2+7L_{\rm NLL3L}^{(-)}+6\right) \tag{151}$$

with $L_{\rm NLL3L}^{(-)}$ and ν being given by equations (140*a*) and (140*b*), and the *new* lower limit NLLN4 is given by (95) with $\sigma = 4g/(e\pi)$.

The other, previously known limits read as follows:

$$\text{BSN:}\quad N < \tfrac{1}{2}g^2(g^2+1) \tag{152}$$

$$\text{Lcentral:}\quad N < 0.864g^3 \tag{153}$$

$$\text{CMSN:}\quad N < 1.698(g^3+2.3562g^2+1.116g+0.1473). \tag{154}$$

The BSN and the CMSN limits can be improved: instead of using the limit on L provided by these limits ($L_{\rm BSL}^{(+)}$ and $L_{\rm CMSL}^{(+)}$), we can use the best upper limits $L_{\rm eff}^{(+)}$ (63); the BSN and CMSN limits obtained in this manner are called here *improved* BSN and CMSN limits:

$$\textit{improved}\ \text{BSN:}\quad N < \frac{1}{2}g^2\left(\frac{4g}{e}+1\right) \tag{155}$$

$$\textit{improved}\ \text{CMSN:}\quad N < 0.5202(g^3+2.179g^2+1.726g+0.4806). \tag{156}$$

Figure 4 presents a comparison between the various limits and the exact result. It shows that the limits on the *total* number of bound states which can be expressed in a neat form are

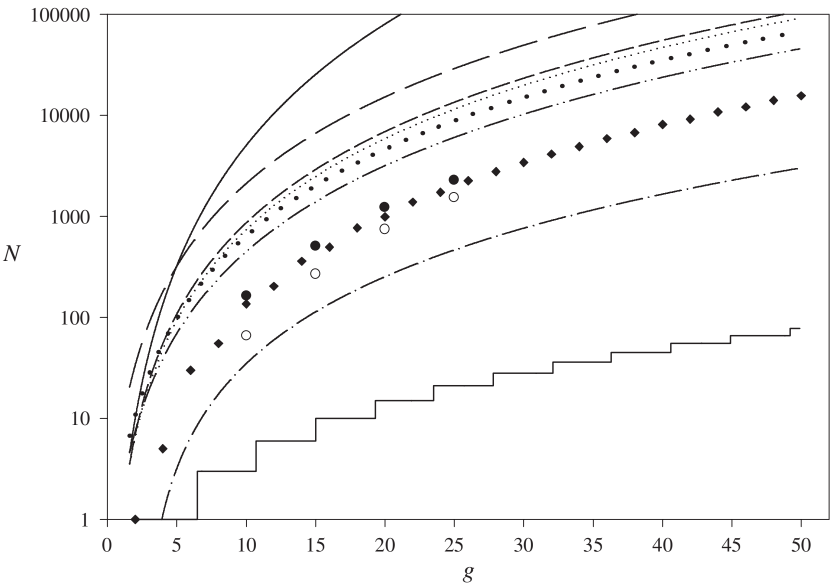

Figure 4. Comparison between the exact value of N (diamonds), the upper limits BSN (152) (solid), CMSN (154) (long-dashed), Lcentral (153) (short-dashed), *improved* BSN (155) (dotted), *improved* CMSN (156) (large dotted), NUL2Nm (150) (dashed-double-dotted) and the lower limits NLLN3 (151) (dashed-dotted) and NLLN4 (95) (ladder) for the E potential (97). The full and the open circles correspond, respectively, to the NUL2N and NLL2N limits.

not very stringent. (Indeed, the best results are yielded by the upper limit NUL2Nm which is obtained using only a limit on the number of S-wave bound states, N_0, and the simple limit $L_{\text{eff}}^{(+)}$ on the maximal value L of ℓ for which bound states do exist.) There are at least three reasons for this. First, most of the limits do not contain the appropriate functional of the potential (as identified by the asymptotic behaviour at large g of N, see (16*b*)): only the Lieb limit Lcentral, see (89), features the correct form, but the numerical factor is not optimal and indeed too large (by approximately a factor of 7). The second reason is that for every value of ℓ, there is a round-off error introduced by the limit; to obtain the limit on the total number of bound states we sum all these errors. The third reason is that to make the summation over the values of ℓ we must have an explicit dependence of the limits on ℓ, and this entails that we cannot use some of the limits we found; in particular, we cannot use the *new* upper and lower limits NUL2 and the NLL2, which are quite stringent, to obtain a neat formula. But we can use them and compute upper and lower limits on N_ℓ, then sum all these contributions to obtain upper and lower limits on N, the sum being stopped when $N_\ell^{(+)}$ is smaller than 1 and $N_\ell^{(-)}$ is negative (see section 1.6). We call NUL2N, respectively NLL2N, the upper, respectively lower, limit on the total number of bound states N obtained (from NUL2, respectively NLL2) via this (inelegant) procedure. Figure 4 shows, for four values of g, that these limits are quite stringent.

The second test is performed with the Y potential. Again, the exact total number of bound states is computed numerically, as indicated above.

The *new* upper limit NUL2Nm, see (92), reads

$$\text{NUL2Nm}: \quad N < \frac{1}{8}\left(\frac{2g}{\sqrt{e}}+1\right)^2\left(2g\sqrt{\frac{2}{\pi}}+\frac{x^2-y^2}{\pi}+\frac{1}{\pi}\log\frac{x}{y}+1\right) \quad (157)$$

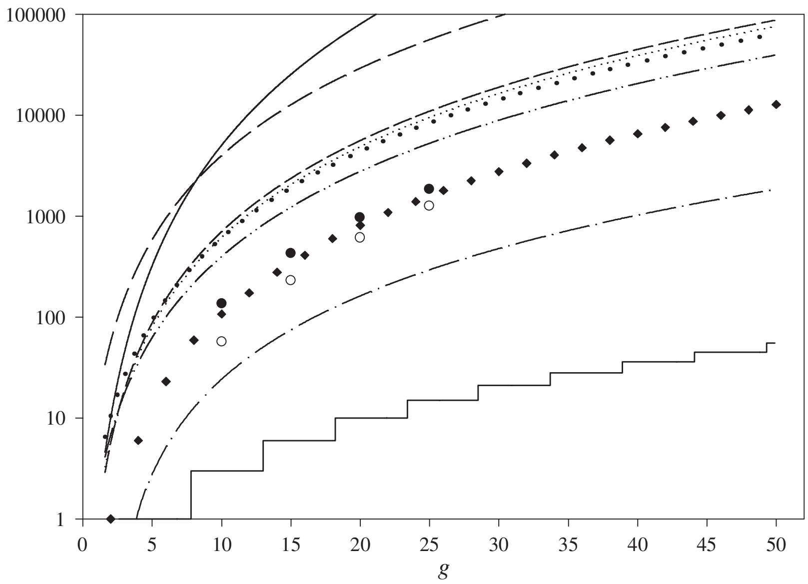

Figure 5. Comparison between the exact value of *N* (diamond), the upper limits BSN (159) (solid), CMSN (161) (long-dashed), Lcentral (160) (short-dashed), *improved* BSN (162) (dotted) and *improved* CMSN (163) (large dotted), NUL2Nm (157) (dashed-double-dotted) and the lower limits NLLN3 (158) (dashed-dotted) and NLLN4 (95) (ladder) for the Y potential (98). The full and the open circles correspond, respectively, to the NUL2N and NLL2N limits.

where y and x are defined by $\mathrm{erf}(y) = \sqrt{\pi/8}/g$, $\mathrm{erf}(x) = 1 - \sqrt{\pi/8}/g$. The *new* limits NLLN3 and NLLN4 are

$$\text{NLLN3:} \quad N > \frac{\nu}{6}\left(2\left(L_{\text{NLL3L}}^{(-)}\right)^2 + 7L_{\text{NLL3L}}^{(-)} + 6\right) \tag{158}$$

with $L_{\text{NLL3L}}^{(-)}$ and ν being given by equations (145) and (145*b*); while the lower limit NLLN4 is given by (95) with $\sigma = 2g/(\sqrt{e}\pi)$.

The other previously known limits read

$$\text{BSN:} \quad N < \tfrac{1}{2}g^2(g^2+1) \tag{159}$$

$$\text{Lcentral:} \quad N < 0.703g^3 \tag{160}$$

$$\text{CMSN:} \quad N < 3.3422(g^3 + 1.88g^2 + 0.711g + 0.075). \tag{161}$$

$$\text{improved BSN:} \quad N < \frac{1}{2}g^2\left(\frac{2g}{\sqrt{e}} + 1\right) \tag{162}$$

$$\text{improved CMSN:} \quad N < 0.4924(g^3 + 2.237g^2 + 1.781g + 0.5078). \tag{163}$$

A comparison between the various limits and the exact numerical results is presented in figure 5, in analogy with the case of the E potential and with analogous conclusions, see above.

3. Proofs

In this section we prove the *new* results reported in section 1. Because we tried and presented those results in section 1 in a user-friendly order, the proofs given below do not follow the same order, due to the need here to follow a more logical sequence. To provide some guidance we divide this section into several subsections, but we must forewarn the reader that a sequential reading is essential to understand what goes on.

3.1. Tools

The starting point of our treatment is the well-known fact (see section 1) that the number N_ℓ of (ℓ-wave) bound states possessed by the potential $V(r)$ coincides with the number of zeros, in the interval $0 < r < \infty$, of the function $u(r)$ uniquely defined (up to an irrelevant multiplicative constant) as the solution of the zero-energy ℓ-wave radial Schrödinger equation

$$u''(r) = \left[V(r) + \frac{\ell(\ell+1)}{r^2}\right] u(r) \tag{164a}$$

with the boundary condition

$$u(0) = 0. \tag{164b}$$

To get an efficient handle on the task of counting these zeros (or rather, of providing upper and lower limits on their number) it is convenient to introduce a new dependent variable $\eta(r)$ related to $u(r)$ as follows:

$$[-U(r)]^{1/2} \cot[\eta(r)] = f(r) + \frac{u'(r)}{u(r)}. \tag{165}$$

Here we moreover introduce two new functions: a function $f(r)$, which we reserve to assign at our convenience below with the only proviso that it be finite in the open interval $0 < r < \infty$, and non-negative throughout this interval,

$$f(r) \geqslant 0 \qquad \text{for} \quad 0 \leqslant r < \infty \tag{166}$$

and a function $U(r)$, which might or might not coincide with the potential $V(r)$ (see below) but that (unless we explicitly state otherwise) we require to be finite in the open interval $0 < r < \infty$, to satisfy (at least) the properties (see (2)–(4))

$$-U(r) = |U(r)| \tag{167}$$

$$\lim_{r\to\infty} [r^{2+\varepsilon} U(r)] = 0 \tag{168}$$

$$\lim_{r\to 0} [r^{2-\varepsilon} U(r)] = 0 \tag{169}$$

and to be related to the potential $V(r)$ as follows:

$$U(r) + W(r) = V(r) + \frac{\ell(\ell+1)}{r^2} \tag{170}$$

where we still reserve the privilege to assign the function $W(r)$ at our convenience (a possibility will be to set $U(r) = V(r)$ hence $W(r) = \ell(\ell+1)/r^2$; but it will not be the only one, see below). Of course the function $\eta(r)$ (as well as $u(r)$) depends on ℓ, although for notational simplicity we omit indicating this explicitly, and this remark may as well apply to the other functions, $f(r)$, $U(r)$, $W(r)$, introduced here and utilized below.

It is then easy to see that the function $\eta(r)$ is uniquely characterized by the first-order nonlinear ODE (implied by (165) with (170), and (164*a*)):

$$\begin{aligned}\eta'(r) = {} & |U(r)|^{1/2} + |U(r)|^{-1/2}\{[f(r)]^2 - f'(r) - W(r)\}\sin^2[\eta(r)] \\ & - \{[4|U(r)|]^{-1}U'(r) + f(r)\}\sin[2\eta(r)]\end{aligned} \tag{171}$$

with the boundary condition (implied by (165) with (164*b*) and (169))

$$\eta(0) = 0. \tag{172}$$

We assume the function $\eta(r)$ to be continuous, disposing thereby of the mod(π) ambiguity entailed by the definition (165).

It is now easy to see that the function $\eta(r)$ provides a convenient tool for evaluating the number of zeros N_ℓ of $u(r)$. Indeed, if we denote with z_n the zeros of $u(r)$, $u(z_n) = 0$, ordered so that

$$0 \equiv z_0 < z_1 < \cdots < z_{N_\ell} \tag{173}$$

(see section 1), since clearly (see (171)) whenever $\eta(r)$ is an integer multiple of π the derivative $\eta'(r)$ is non-negative, $\eta'(r) \geqslant 0$, we may conclude (see (165)) that

$$\eta(z_n) = n\pi \qquad n = 0, 1, \ldots, N_\ell \tag{174}$$

with

$$(n-1)\pi \leqslant \eta(r) \leqslant n\pi \qquad \text{for} \quad z_{n-1} \leqslant r \leqslant z_n \quad n = 1, \ldots, N_\ell. \tag{175}$$

It is moreover plain that, provided (see (165))

$$\lim_{r\to\infty} \left[\frac{|U(r)|^{1/2}}{f(r) + \frac{u'(r)}{u(r)}} \right] = 0 \tag{176}$$

there holds the asymptotic relation

$$\eta(\infty) = N_\ell \pi \tag{177}$$

and that this asymptotic value is approached from above. (To prove the last statement one sets, in the asymptotic large r region, $\eta(r) = N_\ell\pi + \varepsilon(r)$ with $|\varepsilon(r)| \ll 1$, and uses the asymptotic estimates $\sin^2(N_\ell\pi + \varepsilon) \approx \varepsilon^2$, $\sin[2(N_\ell\pi + \varepsilon)] \approx 2\varepsilon$ to rewrite the ODE (171) in the asymptotic region as follows (recall that $U(r)$ is non-positive, see (167)):

$$\varepsilon'(r) \approx [-U(r)]^{1/2} + \left\{ \frac{U'(r)}{2U(r)} - 2f(r) \right\} \varepsilon(r). \tag{178}$$

One can then replace the approximate equality sign $\approx$ in this formula with the equality sign $=$ and integrate the resulting linear ODE, obtaining

$$\varepsilon(r) = |U(r)|^{1/2} \int_R^r \mathrm{d}x \exp\left[2 \int_r^x \mathrm{d}y\, f(y) \right] \tag{179}$$

where R is a (finite) integration constant, and this formula (valid in the asymptotic, large r, region, where $r > R$) shows that $\varepsilon(r)$ is indeed positive).

To get more detailed information on the behaviour of the function $\eta(r)$ we make the additional assumption that the assignments of the auxiliary functions $f(r)$ and $W(r)$ guarantee validity of the following inequality:

$$[f(r)]^2 - f'(r) - W(r) \geqslant 0 \tag{180}$$

(it would be enough for our purposes that this inequality be valid only at large values of r, but for simplicity we assume hereafter its validity for all values of r, $0 \leqslant r < \infty$). It is then clear from the ODE (171) satisfied by $\eta(r)$ that, wherever $\eta(r)$ takes a value which is an odd integer multiple of $\pi/2$, namely at the points $r = b_n$ such that

$$\eta(b_n) = \tfrac{1}{2}(2n-1)\pi \qquad n = 1, 2, \ldots, N_\ell \tag{181}$$

its derivative $\eta'(r)$ is non-negative, $\eta'(b_n) \geqslant 0$. Hence we may complement the information provided by (174) with that provided by formula (181) and moreover replace the information provided by (175) with the following more detailed information:

$$(n-1)\pi \leqslant \eta(r) \leqslant \tfrac{1}{2}(2n-1)\pi \qquad \text{for} \quad z_{n-1} \leqslant r \leqslant b_n \quad n = 1, \ldots, N_\ell \tag{182}$$

$$(2n-1)\frac{\pi}{2} \leqslant \eta(r) \leqslant n\pi \qquad \text{for} \quad b_n \leqslant r \leqslant z_n \quad n = 1, \ldots, N_\ell \tag{183}$$

which also entails that the points z_n and b_n are interlaced,

$$z_0 < b_1 < z_1 < b_2 < \cdots < z_{N_\ell - 1} < b_{N_\ell} < z_{N_\ell} < \infty. \tag{184}$$

And note in particular that these formulae entail the following important inequality (implied by the non-existence of $b_{N_\ell+1}$), valid for all values of r, $0 \leqslant r < \infty$:

$$0 \leqslant \eta(r) < \left(N_\ell + \tfrac{1}{2}\right)\pi. \tag{185}$$

It is moreover clear that the maximum value of $\eta(r)$,

$$\hat{\eta} = \max_{0 \leqslant r < \infty} [\eta(r)] \tag{186}$$

is actually attained in the interval $z_{N_\ell} < r < \infty$, and that it lies in the range

$$N_\ell \pi \leqslant \hat{\eta} < \left(N_\ell + \tfrac{1}{2}\right)\pi. \tag{187}$$

We are now in a position to derive the *new* upper and lower limits reported in section 1.

3.2. Proof of the lower limits NLL4

We begin by proving the lower limit NLL4, see (61). To this end we assign as follows the functions $f(r)$ and $W(r)$:

$$f(r) = \frac{\ell}{r} \qquad W(r) = \frac{\ell(\ell+1)}{r^2} \tag{188}$$

entailing that the left-hand side of the inequality (180) vanishes, that

$$U(r) = V(r) \tag{189}$$

that the definition (165) now reads

$$|V(r)|^{1/2} \cot[\eta(r)] = \frac{\ell}{r} + \frac{u'(r)}{u(r)} \tag{190}$$

and, most importantly, that (171) reads

$$\eta'(r) = |V(r)|^{1/2} - \left\{\frac{V'(r)}{4|V(r)|} + \frac{\ell}{r}\right\} \sin[2\eta(r)]. \tag{191}$$

Here we are assuming the potential $V(r)$ to satisfy condition (2).

Before proceeding with the proof, let us note that, for the potential (34), the second term on the right-hand side of the ODE (191) vanishes, hence one immediately obtains

$$\eta(r) = \frac{g}{2\ell+1} \min\left[\alpha^{2\ell+1}, \left(\frac{r}{R}\right)^{2\ell+1}\right]. \tag{192}$$

Hence (see (177)), for the potential (34),

$$N_\ell = \left\{\left\{\frac{g\alpha^{2\ell+1}}{\pi(2\ell+1)}\right\}\right\} \tag{193}$$

where as usual the double braces signify that the integer part of their contents must be taken. This observation implies that the upper and lower limits which are obtained (see below) by massaging the last term on the right-hand side of the ODE (191) are generally the best possible, being saturated by the potential (34) (if need be, with an appropriate choice of the parameter α, see section 1).

To prove (61) with (8) we now introduce the auxiliary function $\eta_{\mathrm{lo}}(r)$ via the ODE

$$\eta'_{\mathrm{lo}}(r) = |V(r)|^{1/2} - \left\{ \frac{V'(r)}{2|V(r)|} + \frac{2\ell}{r} \right\} \eta_{\mathrm{lo}}(r) \tag{194}$$

with the boundary condition

$$\eta_{\mathrm{lo}}(0) = 0. \tag{195}$$

We then assume the potential $V(r)$ to satisfy, in addition to (2), condition (5). It is then plain (see (191) with (172), (194) with (195), and (5)) that, for all values of r, $0 \leqslant r < \infty$,

$$\eta_{\mathrm{lo}}(r) \leqslant \eta(r). \tag{196}$$

(Indeed (194) is obtained from (191) via the replacement $\sin(x) \Rightarrow x$, and for positive x, $\sin(x) \leqslant x$; hence $\eta_{\mathrm{lo}}(r)$ can never overtake $\eta(r)$ because, at the overtaking point, a comparison of (194) with (191) entails $\eta'_{\mathrm{lo}}(r) \leqslant \eta'(r)$, which negates the possibility of performing the overtaking). But the linear ODE (194) with (195) can be easily integrated to yield (recalling (2))

$$\eta_{\mathrm{lo}}(r) = \frac{r|V(r)|^{1/2}}{2\ell + 1} \tag{197}$$

hence we conclude (see (186), (196), (197) and (8)) that

$$\hat{\eta} \geqslant \frac{\pi\sigma}{2(2\ell + 1)} \tag{198}$$

and via (187) this entails the lower limit NLL4, see (61), which is thereby proved.

3.3. Proof of the upper and lower limits NUL2 and NLL2

Let us now prove the upper and lower limits NUL2 and NLL2, see (44) and (45), on the number N_0 of S-wave bound states possessed by the central potential $V(r)$, under the assumption that this potential satisfies conditions (43). The main tool of the proof is the same function $\eta(r)$ as defined in the preceding subsection 3.2, which is therefore now defined by the formula (see (190))

$$|V(r)|^{1/2} \cot[\eta(r)] = \frac{u'(r)}{u(r)} \tag{199}$$

and satisfies the ODE (see (191))

$$\eta'(r) = |V(r)|^{1/2} - \frac{V'(r)}{4|V(r)|} \sin[2\eta(r)]. \tag{200}$$

Note that this is just the function $\eta(r)$ that provided our main analytical tool in [4]; however, conditions (43a) and (43b) (which clearly imply $V(r_-) = 0$) entail now, via (199), the condition

$$\eta(r_-) = 0 \tag{201}$$

as well as the fact that $u(r)$ is concave in the interval $0 \leqslant r \leqslant r_-$ (see (164a) and (43a)), hence it has no zero in that interval, hence

$$r_- < b_1 < z_1 \tag{202}$$

(see (174), (181) and (199)). Likewise the fact that $u(r)$ is also concave in the interval $r_+ \leqslant r < \infty$ (see (164a) and (43d)), hence it has no extremum in that interval, entails

$$z_{N_0-1} < b_{N_0} < r_+. \tag{203}$$

To obtain the upper limit NUL2 we now integrate the ODE (200) from z_1 to z_{N_0-1} (and note that, thanks to (202) and (203), as well as (43), we can hereafter replace, whenever convenient, $|V(r)|$ with $-V^{(-)}(r)$, see (9)):

$$\eta(z_{N_0-1}) - \eta(z_1) = (N_0-2)\pi = \int_{z_1}^{z_{N_0-1}} \mathrm{d}r[-V^{(-)}(r)]^{1/2} - \frac{1}{4}\int_{z_1}^{r_{\min}} \mathrm{d}r \frac{V'(r)}{|V(r)|}\sin[2\eta(r)] - \frac{1}{4}\int_{r_{\min}}^{z_{N_0-1}} \mathrm{d}r \frac{V'(r)}{|V(r)|}\sin[2\eta(r)]. \tag{204}$$

The first equality is of course entailed by (174). As for the second equation, note that we conveniently split the integration of the second term on the right-hand side of (200) into two parts. The properties (43*b*), (43*c*) (as well as the obvious fact that $|\sin(x)| \leqslant 1$), allow us to majorize the right-hand side of equation (204). We thereby get

$$(N_0-2)\pi < \int_{z_1}^{z_{N_0-1}} \mathrm{d}r[-V^{(-)}(r)]^{1/2} + \frac{1}{4}\int_{z_1}^{r_{\min}} \mathrm{d}r \frac{V'(r)}{V(r)} - \frac{1}{4}\int_{r_{\min}}^{z_{N_0-1}} \mathrm{d}r \frac{V'(r)}{V(r)} \tag{205}$$

hence,

$$(N_0-2)\pi < \int_{z_1}^{z_{N_0-1}} \mathrm{d}r[-V^{(-)}(r)]^{1/2} + \frac{1}{4}\log\left\{\frac{[V^{(-)}(r_{\min})]^2}{V^{(-)}(z_1)V^{(-)}(z_{N_0-1})}\right\}. \tag{206}$$

We now need to find quantities p and q, defined only in terms of the potential, such that $p \leqslant z_1$ and $q \geqslant z_{N_0-1}$ and also such that $|V(p)| \leqslant |V(z_1)|$ and $|V(q)| \leqslant |V(z_{N_0-1})|$. Let us first consider the 'favourable' case (which is obtained for a sufficiently attractive potential): $z_1 \leqslant r_{\min}$ and $z_{N_0-1} \geqslant r_{\min}$. In this case, we integrate (200) from b_1 to z_1 and since in this interval both $V'(r)/|V(r)|$ and $\sin[2\eta(r)]$ are *negative*, we infer

$$\eta(z_1) - \eta(b_1) = \frac{\pi}{2} \leqslant \int_{b_1}^{z_1} \mathrm{d}r[-V^{(-)}(r)]^{1/2} \tag{207}$$

hence *a fortiori* (see (202))

$$\int_0^{z_1} \mathrm{d}r[-V^{(-)}(r)]^{1/2} > \frac{\pi}{2}. \tag{208}$$

If we define p via the formula

$$\int_0^{p} \mathrm{d}r[-V^{(-)}(r)]^{1/2} = \frac{\pi}{2} \tag{209}$$

then we conclude (by comparing (208) with (209)) that $p < z_1$. Moreover, since we have supposed $z_1 \leqslant r_{\min}$, we have also $|V(p)| < |V(z_1)|$. We then integrate (200) from z_{N_0-1} to b_{N_0}, and taking advantage of the fact that in this interval both $V'(r)/|V(r)|$ and $\sin[2\eta(r)]$ are *positive*, we infer

$$\eta(b_{N_0}) - \eta(z_{N_0-1}) = \frac{\pi}{2} \leqslant \int_{z_{N_0-1}}^{b_{N_0}} \mathrm{d}r[-V^{(-)}(r)]^{1/2} \tag{210}$$

hence *a fortiori* (see (203))

$$\frac{\pi}{2} < \int_{z_{N_0-1}}^{\infty} \mathrm{d}r[-V^{(-)}(r)]^{1/2}. \tag{211}$$

Analogously, if we define q via the formula

$$\int_q^{\infty} \mathrm{d}r[-V^{(-)}(r)]^{1/2} = \frac{\pi}{2} \tag{212}$$

we conclude that $q > z_{N_0-1}$. Moreover, since we have supposed $z_{N_0-1} \geqslant r_{\min}$, we have also $|V(q)| < |V(z_{N_0-1})|$. Thus if these two relations, $z_1 \leqslant r_{\min}$ and $z_{N_0-1} \geqslant r_{\min}$, hold, we obtain

$$N_0 < \frac{1}{\pi}\int_0^\infty \mathrm{d}r[-V^{(-)}(r)]^{1/2} + \frac{1}{4\pi}\log\left\{\frac{[V^{(-)}(r_{\min})]^2}{V^{(-)}(p)V^{(-)}(q)}\right\} + 1 \tag{213}$$

where we have used equations (209) and (212). Relation (213) implies the validity of relation (44).

We now need to consider the cases where $z_1 > r_{\min}$ or $z_{N-1} < r_{\min}$. In these cases we could have, for example, $p < z_1$ and $|V(p)| > |V(z_1)|$.

Firstly, suppose $z_1 > r_{\min}$ which implies that $z_{N_0-1} > r_{\min}$. Now we impose a first condition for the applicability of the limit NUL2: $p \leqslant r_{\min}$. This condition is always true for a potential which possesses enough bound states; in practice, the limit NUL2 will be applicable only when the attractive strength of the potential is large enough. This condition ensures that $p < z_1$ (this is not necessarily true, with the definition (209), when $z_1 > r_{\min}$). Indeed, if $z_1 \leqslant r_{\min}$, we have proved it above, and if $z_1 > r_{\min}$, this is still true since $p \leqslant r_{\min}$. Moreover, since $z_1 > r_{\min}$, we have $|V(z_1)| > |V(z_{N_0-1})| > |V(q)| \geqslant M$, with

$$M = \min(|V(p)|, |V(q)|) = \min[-V^{(-)}(p), -V^{(-)}(q)]. \tag{214}$$

This implies the validity of relation (44).

Secondly, suppose $z_{N_0-1} < r_{\min}$ which obviously implies $z_1 < r_{\min}$. Now we impose a second condition for the applicability of the limit: $q \geqslant r_{\min}$. This condition is always true for a potential which possesses enough bound states. This condition ensures that $q > z_{N_0-1}$ (this is not necessarily true, with the definition (212), when $z_{N_0-1} < r_{\min}$). Indeed, if $z_{N_0-1} \geqslant r_{\min}$, we have proved it above, and if $z_{N_0-1} < r_{\min}$, this is still true since $q \geqslant r_{\min}$. Moreover, since $z_{N_0-1} < r_{\min}$, we have $\left|V\left(z_{N_0-1}\right)\right| > |V(z_1)| > |V(p)| \geqslant M$, with M defined again by (214). This implies, via the definitions (7) and (9), the validity of relation (44), and concludes our proof of the *new* upper limit NUL2.

The proof of the *new* lower limit NLL2, see (45), is completely analogous, except that one integrates the ODE (200) from p to q

$$\eta(q) - \eta(p) \geqslant \int_p^q \mathrm{d}r[-V^{(-)}(r)]^{1/2} - \frac{1}{4}\int_p^{r_{\min}} \mathrm{d}r\frac{V'(r)}{V(r)} + \frac{1}{4}\int_{r_{\min}}^q \mathrm{d}r\frac{V'(r)}{V(r)} \tag{215}$$

and from the inequalities (actually valid for any positive radius, see (185))

$$\eta(p) \geqslant 0 \tag{216}$$

$$\eta(q) < \left(N_0 + \tfrac{1}{2}\right)\pi \tag{217}$$

we clearly infer

$$\eta(q) - \eta(p) < \left(N_0 + \tfrac{1}{2}\right)\pi. \tag{218}$$

Hence

$$\left(N_0 + \frac{1}{2}\right)\pi > \int_p^q \mathrm{d}r[-V^{(-)}(r)]^{1/2} - \frac{1}{4}\int_p^{r_{\min}} \mathrm{d}r\frac{V'(r)}{V(r)} + \frac{1}{4}\int_{r_{\min}}^q \mathrm{d}r\frac{V'(r)}{V(r)} \tag{219}$$

hence, via the definitions (209) and (212) of p and q,

$$\left(N_0 + \frac{3}{2}\right)\pi > \int_0^\infty \mathrm{d}r[-V^{(-)}(r)]^{1/2} - \frac{1}{4}\log\left\{\frac{[V^{(-)}(r_{\min})]^2}{V^{(-)}(p)V^{(-)}(q)}\right\}. \tag{220}$$

Note that this inequality is true in any case, provided $p \leqslant r_{\min} \leqslant q$, and it implies (again, via the definitions (7) and (9), as well as (214)) the validity of the marginally less stringent lower bound (45) (we preferred to present in section 1 the lower bound (45) rather than the more stringent one implied by (220) to underline its analogy with the upper bound (44)).

3.4. Proof of the lower limits NLL3s and NLL3

Let us now proceed and prove (following [4]) the *new* lower limits NLL3s and NLL3, see (57) and (59). The proof is analogous to the proofs of the NUL2 and NLL2 limits given in the previous subsection except that now instead of considering equation (200) for $\eta(r)$ we use equation (191). To obtain NLL3s, we integrate the ODE (191) from p to an arbitrary radius $s \geqslant r_{\min}$:

$$\eta(s) - \eta(p) = \int_p^s \mathrm{d}r[-V^{(-)}(r)]^{1/2} - \frac{1}{4}\int_p^{r_{\min}} \mathrm{d}r \frac{V'(r)}{|V(r)|}\sin[2\eta(r)] \\ - \frac{1}{4}\int_{r_{\min}}^s \mathrm{d}r\frac{V'(r)}{|V(r)|}\sin[2\eta(r)] - \int_p^s \mathrm{d}r\frac{\ell}{r}\sin[2\eta(r)]. \tag{221}$$

The right-hand side of this equation can be minorized (since $\eta(s) < (N_\ell + 1/2)\pi$ and $\eta(p) > 0$, see (185)) to yield

$$\left(N_\ell + \frac{1}{2}\right)\pi > \eta(s) - \eta(p) \geqslant \int_p^s \mathrm{d}r[-V^{(-)}(r)]^{1/2} \\ - \frac{1}{4}\int_p^{r_{\min}} \mathrm{d}r\frac{V'(r)}{V(r)} + \frac{1}{4}\int_{r_{\min}}^s \mathrm{d}r\frac{V'(r)}{V(r)} - \ell\log\left(\frac{s}{p}\right). \tag{222}$$

From the definition of p, see (47), we finally obtain

$$(N_\ell + 1)\pi > \int_0^s \mathrm{d}r[-V^{(-)}(r)]^{1/2} - \frac{1}{4}\log\left\{\frac{[V^{(-)}(r_{\min})]^2}{V^{(-)}(p)V^{(-)}(s)}\right\} - \ell\log\left(\frac{s}{p}\right) \tag{223}$$

which coincides with the lower limit NLL3s, see (57), which is thereby proved.

To get the lower limit NLL3 we proceed as above, except that we integrate from p to q, see (47) and (48).

3.5. Proof of the results in terms of comparison potentials (see section 1.4)

Let us now prove relations (65) and (66). We assume for this purpose that the potential $V(r)$ satisfies the negativity condition (2), but we require no monotonicity condition on $V(r)$; we do however require the potential $V(r)$ to be nonsingular for $0 \leqslant r < \infty$ and to satisfy the conditions, see (3) and (4), that are sufficient to guarantee that the quantity S, see (7), be finite.

Let us now replace the potential $V(r)$ with $V(r) + H_\lambda^{(\ell)}(r)$, so that the radial Schrödinger equation, see (164*a*), now reads

$$u''(r) = \left[V(r) + H_\lambda^{(\ell)}(r) + \frac{\ell(\ell+1)}{r^2}\right]u(r) \tag{224}$$

and relation (170) now reads

$$U(r) + W(r) = V(r) + H_\lambda^{(\ell)}(r) + \frac{\ell(\ell+1)}{r^2}. \tag{225}$$

Let us moreover set (see (165))

$$U(r) = V(r) \tag{226}$$

$$f(r) = \frac{V'(r)}{4V(r)} \tag{227}$$

so that (see (225))

$$W(r) = H_\lambda^{(\ell)}(r) + \frac{\ell(\ell+1)}{r^2} \tag{228}$$

with the additional requirement

$$H_\lambda^{(\ell)}(r) + \frac{\ell(\ell+1)}{r^2} - [f(r)]^2 + f'(r) = \beta|V(r)| \tag{229}$$

where $\beta < 1$. (Note that we imposed in (180) that the left-hand side of (229) be positive. Actually this restriction, which was introduced to prove that $\eta'(b_n) \geqslant 0$ (see (181)), was too strong for our needs. Indeed, one can verify from (171) that we still have $\eta'(b_n) \geqslant 0$ provided $\beta < 1$). It is easily seen that this entails for $H_\lambda^{(\ell)}(r)$ the definition (64) via the assignment

$$\beta = 1 - 4\lambda^2. \tag{230}$$

Hence these assignments imply that the definition of $\eta(r)$, see (165), now reads

$$[-V(r)]^{1/2}\cot[\eta(r)] = \frac{V'(r)}{4V(r)} + \frac{u'(r)}{u(r)} \tag{231}$$

and, most importantly, that equation (171) satisfied by $\eta(r)$ now becomes simply

$$\eta'(r) = |V(r)|^{1/2}\{1 - \beta\sin^2[\eta(r)]\} \tag{232}$$

entailing

$$\eta'(r)\{1 - \beta\sin^2[\eta(r)]\}^{-1} = |V(r)|^{1/2}. \tag{233}$$

Both sides of the last equation are now easily integrated, the right-hand side from $r = 0$ to $r = \infty$, and the left-hand side, correspondingly, from $\eta = 0$ (see (172), which is clearly implied by (213) and by (164*b*)) to $\eta(\infty)$, yielding (see (7) and (230))

$$\eta(\infty) = \lambda S\pi. \tag{234}$$

It is, on the other hand, clear that in this case as well

$$N_\ell\pi \leqslant \eta(\infty) < (N_\ell + 1)\pi. \tag{235}$$

(Indeed, while in this case relation (176) does not hold and therefore neither (177) nor (187) need be true, relation (174) is still implied by the definition (231), and moreover (233) clearly implies $\eta'(z_n) \geqslant 0$, entailing validity of these inequalities). Hence (see (234))

$$N_\ell^{(V+H_\lambda^{(\ell)})} = \{\{\lambda S\}\} \tag{236}$$

where as usual the double braces denote the integer part. In this formula the notation $N_\ell^{(V+H_\lambda^{(\ell)})}$ denotes the number of ℓ-wave bound states possessed by the potential $V(r) + H_\lambda^{(\ell)}(r)$.

But if the 'additional potential' $H_\lambda^{(\ell)}(r)$, see (64), is nowhere negative, this potential $V(r) + H_\lambda^{(\ell)}(r)$ cannot possess fewer (ℓ -wave) bound states than the potential $V(r)$, hence the lower limit (65) is proved. And under the same conditions, if the function $H_\lambda^{(\ell)}(r)$ is nowhere positive, the potential $V(r) + H_\lambda^{(\ell)}(r)$ does not have fewer bound states than the potential $V(r)$, hence the upper limit (66) is proved.

3.6. Proof of the upper and lower limits NUL1 and NLL1

Next, we prove the *new* upper and lower limits NUL1 and NLL1, see (37) and (38). To prove them we assume the potential $V(r)$ to possess the properties (36), and we set $\ell = 0$. We moreover set (see (165))

$$f(r) = 0 \tag{237}$$

and

$$U(r) = -a^2 \tag{238}$$

where a is a *positive* constant, $a > 0$, the value of which we reserve to assign at our convenience later. Note that in this case our assignment for $U(r)$ does not satisfy conditions (168) and (169), and that the definition (165) of $\eta(r)$ now reads

$$a \cot[\eta(r)] = \frac{u'(r)}{u(r)}. \tag{239}$$

Consistent with these assignments we also set (see (170))

$$W(r) = V(r) + a^2 \tag{240}$$

and the equation satisfied by $\eta(r)$, see (171), now reads

$$\eta'(r) = a \cos^2[\eta(r)] - a^{-1} V(r) \sin^2[\eta(r)]. \tag{241}$$

We then integrate this ODE from $r = r_-$ to $r = r_+$:

$$\eta(r_+) - \eta(r_-) = \int_{r_-}^{r_+} \mathrm{d}r \{a \cos^2[\eta(r)] + a^{-1} |V(r)| \sin^2[\eta(r)]\}. \tag{242}$$

Note that we used (36*b*).

Now the definition (239) of $\eta(r)$ entails that the radii b_n, see (181), coincide with the extrema of the zero-energy wavefunction $u(r)$, $u'(b_n) = 0$. We therefore can use (36*a*) to conclude that, since the zero-energy wavefunction $u(r)$ is concave in the interval $0 \leqslant r < r_-$, the first extremum b_1 must occur after r_-, $r_- < b_1$, hence (see (181))

$$0 \leqslant \eta(r_-) < \frac{\pi}{2}. \tag{243}$$

Likewise, (36*c*) entails that $u(r)$ is concave in the interval $r_+ < r < \infty$, hence the last extremum, b_{N_0}, must occur before r_+, $b_{N_0} < r_+$, while $\eta(r)$ can never reach $(N_0 + 1)\pi$ (note that in this case condition (176) does not hold, hence the more stringent condition (185) does not apply). Hence

$$\left(N_0 - \tfrac{1}{2}\right) \pi < \eta(r_+) < (N_0 + 1)\pi \tag{244}$$

where we are denoting as N_0 the number of bound states possessed by the potential $V(r)$. Hence we may assert that

$$(N_0 - 1)\pi < \eta(r_+) - \eta(r_-) < (N_0 + 1)\pi. \tag{245}$$

From the first of these two inequalities, and (242), we immediately get

$$(N_0 - 1)\pi < \int_{r_-}^{r_+} \mathrm{d}r \{a + a^{-1} |V(r)|\} \tag{246}$$

since the replacement of $\cos^2[\eta(r)]$ and $\sin^2[\eta(r)]$ by unity on the right-hand side of (242) entails a (further) majorization. Hence

$$N_0 < 1 + \frac{1}{\pi} \left[a(r_+ - r_-) + a^{-1} \int_0^\infty \mathrm{d}r [-V^{(-)}(r)] \right] \tag{247}$$

and by setting

$$a = (r_+ - r_-)^{-1/2} \left(\int_0^\infty \mathrm{d}r [-V^{(-)}(r)] \right)^{1/2} \tag{248}$$

we get

$$N_0 < 1 + \frac{2}{\pi} \left[(r_+ - r_-) \int_0^\infty \mathrm{d}r [-V^{(-)}(r)] \right]^{1/2}. \tag{249}$$

The result NUL1, see (37), is thereby proved.

To prove the lower limit NLL1, see (38), we use the second inequality of (245) to get from (242) the inequality

$$(N_0+1)\pi > \eta(r_+) - \eta(r_-) \geqslant \int_0^\infty \mathrm{d}r \, \min[a, a^{-1}[-V^{(-)}(r)]] \tag{250}$$

and this coincides with formula (38). The lower limit NLL1 is thereby proved.

3.7. Unified derivation of the upper limits NUL1 and NUL2

In this subsection we indicate how the derivations of the two upper limits, NUL1 and NUL2, can be unified. This entails that, in this context, the limit NUL2 is the optimal one. In view of the previous developments our treatment here is rather brief.

The starting point of the treatment is the ODE

$$\eta'(r) = |U(r)|^{1/2}\cos^2[\eta(r)] - V(r)|U(r)|^{-1/2}\sin^2[\eta(r)] + \frac{U'(r)}{4U(r)}\sin[2\eta(r)] \tag{251}$$

that corresponds to (171) with $\ell = 0$, $f(r) = 0$, and $W(r) = V(r) - U(r)$, where we always assume $U(r)$ to be non-positive, see (167), but otherwise we maintain the option to assign it at our convenience. Clearly this formula entails

$$\eta'(r) \leqslant \max[|U(r)|^{1/2}, -V(r)|U(r)|^{-1/2}] + \left|\frac{U'(r)}{4U(r)}\right| \tag{252}$$

hence

$$\eta(r_2) - \eta(r_1) \leqslant \int_{r_1}^{r_2} \mathrm{d}r \left\{\max[|U(r)|^{1/2}, -V(r)|U(r)|^{-1/2}] + \left|\frac{U'(r)}{4U(r)}\right|\right\}. \tag{253}$$

It is now clear that two assignments of $U(r)$ recommend themselves. One possibility is to assume that $U(r)$ is constant, implying that the last term on the right-hand side of this inequality vanishes: this is indeed the choice (238), and it leads to the neat upper limit NUL1, see the preceding subsection 3.6. The other, optimal, possibility is to equate the two arguments of the maximum functional on the right-hand side of inequality (253), namely to make the assignment (189), and then to proceed as in subsection 3.4, arriving thereby at the upper limit NUL2 (and note that a closely analogous procedure was used in [4], albeit in the simpler context of a monotonically increasing potential).

3.8. Proof of the upper and lower limits of second kind (see section 1.5)

Let us now prove the results of section 1.5, beginning with the proof of the upper limit (76). To this end let us assume first of all that $r^{(\mathrm{up,decr})}_{J^{(\mathrm{up,decr})}} > 0$ (see (73) and (75)), and then introduce the piecewise constant comparison potential $V^{(+)}(r)$, defined as follows:

$$V^{(+)}(r) = 0 \qquad \text{for} \quad 0 \leqslant r < r^{(\mathrm{up,decr})}_{J^{(\mathrm{up,decr})}} \tag{254a}$$

$$V^{(+)}(r) = V\big(r^{(\mathrm{up,decr})}_{j-1}\big) \qquad \text{for} \quad r^{(\mathrm{up,decr})}_{j-1} \leqslant r < r^{(\mathrm{up,decr})}_{j} \qquad \text{with} \quad j = J^{(\mathrm{up,decr})}, J^{(\mathrm{up,decr})}-1, \ldots, 0 \tag{254b}$$

$$V^{(+)}(r) = V\big(r^{(\mathrm{up,incr})}_{j}\big) \qquad \text{for} \quad r^{(\mathrm{up,incr})}_{j-1} \leqslant r < r^{(\mathrm{up,incr})}_{j} \qquad \text{with} \quad j = 1, \ldots, J^{(\mathrm{up,incr})} \tag{254c}$$

$$V^{(+)}(r) = 0 \qquad \text{for} \quad r_{J^{(\mathrm{up,incr})}} \leqslant r. \tag{254d}$$

It is obvious by construction (if in doubt, draw a graph!) that

$$V(r) \geqslant V^{(+)}(r) \tag{255}$$

hence, if we indicate with $N_0^{(+)}$ the number of S-wave bound states possessed by the potential $V^{(+)}(r)$ (and with N_0 the number of S-wave bound states possessed by the potential $V(r)$), clearly

$$N_0 \leqslant N_0^{(+)}. \tag{256}$$

It is moreover clear from the previous treatment, see in particular (199) and (200), that we now write, in self-evident notation, as follows,

$$|V^{(+)}(r)|^{1/2}\cot[\eta^{(+)}(r)] = \frac{u^{(+)\prime}(r)}{u^{(+)}(r)} \tag{257}$$

$$\eta^{(+)\prime}(r) = |V^{(+)}(r)|^{1/2} - \frac{V^{(+)\prime}(r)}{4|V^{(+)}(r)|}\sin[2\eta^{(+)}(r)] \tag{258}$$

that the number $N_0^{(+)}$ of S-wave bound states possessed by the potential $V^{(+)}(r)$ can be obtained in the usual manner via the solution $\eta^{(+)}(r)$, for $r \geqslant r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}$, of the ODE (258) characterized by the boundary condition (see (257) and (254*a*) entailing $u(r) = u'(0)r$ for $0 \leqslant r \leqslant r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}$)

$$\left|V^{(+)}\left(r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}\right)\right|^{1/2}\cot\left[\eta^{(+)}\left(r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}\right)\right] = \left[r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}\right]^{-1} \tag{259}$$

namely

$$\tan\left[\eta^{(+)}\left(r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}\right)\right] = \left[r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}\right]\left|V^{(+)}\left(r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}\right)\right|^{1/2} \tag{260}$$

entailing

$$\eta^{(+)}\left(r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}\right) < \frac{\pi}{2}. \tag{261}$$

The number $N_0^{(+)}$ of S-wave bound states possessed by the potential $V^{(+)}(r)$ is then characterized by the inequality

$$N_0^{(+)} \leqslant \frac{1}{\pi}\eta^{(+)}\left(r_{J(\mathrm{up,incr})}^{(\mathrm{up,incr})}\right) + \frac{1}{2}. \tag{262}$$

(Indeed, in self-evident notation, $b_{N_0^{(+)}}^{(+)} \leqslant r_{J(\mathrm{up,incr})}^{(\mathrm{up,incr})}$, see (254*d*), hence $\eta^{(+)}\left(b_{N_0^{(+)}}^{(+)}\right) \leqslant \eta^{(+)}\left(r_{J(\mathrm{up,incr})}^{(\mathrm{up,incr})}\right)$ (see (258) and (254*c*)), and $\eta^{(+)}\left(b_{N_0^{(+)}}^{(+)}\right) = \left(N_0^{(+)} - \frac{1}{2}\right)\pi$, see (181)).

We now introduce another solution, $\eta^{(++)}(r)$, of the ODE (258) in the interval $r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})} \leqslant r \leqslant r_{J(\mathrm{up,incr})}^{(\mathrm{up,incr})}$, characterized by the boundary condition

$$\eta^{(++)}\left(r_{J(\mathrm{up,decr})}^{(\mathrm{up,decr})}\right) = \frac{\pi}{2}. \tag{263}$$

It is then plain, from a comparison of the 'initial condition' (263) with (261), that throughout this interval $\eta^{(++)}(r) > \eta^{(+)}(r)$, hence in particular $\eta^{(++)}\left(r_{J(\mathrm{up,incr})}^{(\mathrm{up,incr})}\right) > \eta^{(+)}\left(r_{J(\mathrm{up,incr})}^{(\mathrm{up,incr})}\right)$, hence from (262) we infer *a fortiori*

$$N_0^{(+)} < \frac{1}{\pi}\eta^{(++)}\left(r_{J(\mathrm{up,incr})}^{(\mathrm{up,incr})}\right) + \frac{1}{2}. \tag{264}$$

But the function $\eta^{(++)}(r)$ can be easily evaluated in closed form, since *de facto* it satisfies the ODE

$$\eta^{(++)\prime}(r) = |V^{(+)}(r)|^{1/2}. \tag{265}$$

Indeed the second term on the right-hand side of the ODE (258) now vanishes: inside the intervals in which the potential $V^{(+)}(r)$ is constant, see (254), because its derivative $V^{(+)\prime}(r)$ vanishes, and at the boundary of these intervals, where the potential $V^{(+)}(r)$ is discontinuous, hence its derivative $V^{(+)'}(r)$ features a delta-function contribution, because the term $\sin[2\eta^{(++)}(r)]$ vanishes: indeed, as can be immediately verified, the ODE (265) with the initial condition (263) and the piecewise potential (254), entail that at these boundaries, say at $r = r_n^{(+)}$ with

$$r_1^{(+)} = r_{J^{(\mathrm{up,decr})}}^{(\mathrm{up,decr})} \qquad r_2^{(+)} = r_{J^{(\mathrm{up,decr})}-1}^{(\mathrm{up,decr})}, \ldots, r_{J^{(\mathrm{up,decr})}}^{(+)} = r_1^{(\mathrm{up,decr})}$$
$$r_{J^{(\mathrm{up,decr})}+1}^{(+)} = r_{\min} \qquad r_{J^{(\mathrm{up,decr})}+2}^{(+)} = r_1^{(\mathrm{up,incr})} \qquad r_{J^{(\mathrm{up,decr})}+3}^{(+)} = r_2^{(\mathrm{up,incr})}, \ldots,$$
$$r_{J^{(\mathrm{up,decr})}+J^{(\mathrm{up,incr})}}^{(+)} = r_{J^{(\mathrm{up,incr})}-1}^{(\mathrm{up,incr})} \qquad r_{J^{(\mathrm{up,decr})}+J^{(\mathrm{up,incr})}+1}^{(+)} = r_{J^{(\mathrm{up,incr})}}^{(\mathrm{up,incr})} \tag{266}$$

there hold the relations

$$\eta^{(++)}\left(r_n^{(+)}\right) = n\frac{\pi}{2} \qquad n = 1, 2, \ldots, J^{(\mathrm{up,decr})} + J^{(\mathrm{up,incr})} + 1. \tag{267}$$

This formula entails indeed $\sin\left[2\eta^{(++)}\left(r_n^{(+)}\right)\right] = 0$, and moreover

$$\eta^{(++)}\left(r_{J^{(\mathrm{up,incr})}}^{(\mathrm{up,incr})}\right) = \left(J^{(\mathrm{up,decr})} + J^{(\mathrm{up,incr})} + 1\right)\frac{\pi}{2} \tag{268}$$

hence, via (264),

$$N_0 < \tfrac{1}{2}\left(J^{(\mathrm{up,incr})} + J^{(\mathrm{up,decr})} + 2\right) \tag{269}$$

consistent with the *new* upper limit of the second kind, see (76), in the case $r_{J^{(\mathrm{up,decr})}}^{(\mathrm{up,decr})} > 0$.

If instead $r_{J^{(\mathrm{up,decr})}}^{(\mathrm{up,decr})} \leqslant 0$, the proof is analogous, except that formula (254*a*) is now irrelevant, the 'initial condition' (261) is replaced by

$$\eta^{(+)}(0) = 0 \tag{270}$$

the 'initial condition' (263) is replaced by

$$\eta^{(++)}(0) = \frac{\pi}{2} - \frac{\left|V\left(r_{J^{(\mathrm{up,decr})}-1}^{(\mathrm{up,decr})}\right)\right|^{-1/2}}{r_{J^{(\mathrm{up,decr})}-1}^{(\mathrm{up,decr})}} \tag{271}$$

in the definition (266) of the radii $r_n^{(+)}$ the lower index on the right-hand side is always decreased by one unit (i.e., $r_1^{(+)} = r_{J^{(\mathrm{up,decr})}-1}^{(\mathrm{up,decr})}$ and so on, entailing $r_{J^{(\mathrm{up,decr})}+J^{(\mathrm{up,incr})}}^{(+)} = r_{J^{(\mathrm{up,incr})}}^{(\mathrm{up,incr})}$), hence (268) reads

$$\eta^{(++)}\left(r_{J^{(\mathrm{up,incr})}}^{(\mathrm{up,incr})}\right) = \left(J^{(\mathrm{up,decr})} + J^{(\mathrm{up,incr})}\right)\frac{\pi}{2} \tag{272}$$

and consequently (269) reads

$$N_0 < \tfrac{1}{2}\left(J^{(\mathrm{up,incr})} + J^{(\mathrm{up,decr})} + 1\right) \tag{273}$$

consistent with (76) in the case $r_{J^{(\mathrm{up,decr})}}^{(\mathrm{up,decr})} \leqslant 0$. The proof of the *new* upper limit of the second kind (76) is thereby completed.

The proof of the *new* lower limit of the second kind, see (81), is analogous, and we therefore only outline it here. It is again based on the construction of a piecewise potential $V^{(-)}(r)$ that (in this case) maximizes the potential $V(r)$ and for which the number $N_0^{(-)}$ of S-wave bound states (or rather, an upper limit to this number) can be computed easily in closed form. We construct this piecewise potential $V^{(-)}(r)$ according to the following prescriptions:

$$V^{(-)}(r) = \infty \qquad \text{for} \quad 0 \leqslant r < r_0^{(\text{lo,incr})} \tag{274a}$$

$$V^{(-)}(r) = V\left(r_{j-1}^{(\text{lo,incr})}\right) \qquad \text{for} \quad r_{j-1}^{(\text{lo,incr})} \leqslant r < r_j^{(\text{lo,incr})} \quad \text{with} \quad j = 1, 2, \ldots, J^{(\text{lo,incr})} - 1 \tag{274b}$$

$$V^{(-)}(r) = \max\left[V\left(r_{J^{(\text{lo,incr})}-1}^{(\text{lo,incr})}\right), V\left(r_{J^{(\text{lo,decr})}-1}^{(\text{lo,decr})}\right)\right] \qquad \text{for} \quad r_{J^{(\text{lo,incr})}-1}^{(\text{lo,incr})} \leqslant r < r_{J^{(\text{lo,decr})}-1}^{(\text{lo,decr})} \tag{274c}$$

$$V^{(-)}(r) = V\left(r_j^{(\text{lo,decr})}\right) \qquad \text{for} \quad r_{j+1}^{(\text{lo,decr})} \leqslant r < r_j^{(\text{lo,decr})} \quad \text{with} \quad j = J^{(\text{lo,decr})} - 1 \quad J^{(\text{lo,decr})} - 2, \ldots, 1, 0 \tag{274d}$$

$$V^{(-)}(r) = \max_{r > r_+}[V(r)] \qquad \text{for} \quad r_0^{(\text{lo,decr})} \leqslant r. \tag{274e}$$

It is then obvious (draw graph if in doubt!) that this potential maximizes the original potential $V(r)$,

$$V(r) \leqslant V^{(-)}(r) \tag{275}$$

hence that the number $N_0^{(-)}$ of its bound states provides a lower limit to the number N_0 of bound states of the potential $V(r)$,

$$N_0 \geqslant N_0^{(-)}. \tag{276}$$

But it is also clear, on the basis of the analysis given above (and leaving to the alert reader the task of providing the details required to turn this argument into a rigorous proof), that each of the $J^{(\text{lo,incr})} + J^{(\text{lo,decr})}$ intervals in which the piecewise potential $V^{(-)}(r)$ is negative, see (274*b*), (274*c*) and (274*d*), can accommodate 'half a bound state' (namely it yields an increase by $\pi/2$ of the relevant function $\eta(r)$), except possibly for the central interval around $r_{\min}$, which can or cannot accommodate such 'half a bound state' depending on whether the product of the square root of the modulus of the potential $V^{(-)}(r)$ in that interval times the length of that interval,

$$\left[r_{J^{(\text{lo,decr})}-1}^{(\text{lo,decr})} - r_{J^{(\text{lo,incr})}-1}^{(\text{lo,incr})}\right] \min\left[\left|V\left(r_{J^{(\text{lo,incr})}-1}^{(\text{lo,incr})}\right)\right|^{1/2}, \left|V\left(r_{J^{(\text{lo,decr})}-1}^{(\text{lo,decr})}\right)\right|^{1/2}\right] \tag{277}$$

does or does not amount to no less than $\pi/2$ (and it is easy to verify that the definition of H as given after (81) entails $H = 0$ in the former case, $H = 1$ in the latter: see (77) and (78)). This justifies the expression on the right-hand side of (81), except for the additional term -1 appearing there, which takes care of the two extremal intervals of the potential $V^{(-)}(r)$, see (274*a*) and (274*e*), each of which can at most 'unbound half a bound state' (i.e., cause a decrease of the relevant $\eta(r)$ by $\pi/2$).

4. Outlook

With modern personal computers the calculation of the number of bound states in a given central potential $V(r)$ is an easy numerical task (especially using (191) with (172) and (177)), as well as the numerical computation of the corresponding binding energies and eigenfunctions. It remains however of interest to obtain neat formulae which provide directly in terms of the potential $V(r)$ upper and lower limits for these physical quantities, as indeed demonstrated by the continued attention given to these problems in the recent literature [3–5, 10–12, 14–17]. The technique used in this paper, and in the one that preceded it [4], goes back to the 1960s (see for instance [7]), yet our findings demonstrate that it can still yield remarkably neat, and cogent, *new* results. We plan to explore in the future the applicability of this approach [7] to establish upper and lower limits to the energies of bound states, as well as to obtain upper and lower limits on the number of bound states (or, more generally, of discrete eigenvalues) in more general contexts, including those spectral problems in the context that are relevant for the investigation of integrable nonlinear partial differential equations, a context in which the number of discrete eigenvalues is generally related to the number of solitons, see for instance [8].

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